**A behavior or an experience: the case of the tangent to a cubic polynomial**

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Most students perform mathematical activity for years, never grasping its uniqueness. This often affects their mathematical 'functioning', manifested by defective grasp, misunderstanding, confusion and aversion. Math is viewed by many students as a field of knowledge with rigid demands and excessive, uneventful drill. It is therefore imperative that we as teachers consider the meaning of the cognitive-emotional balance as part of our pedagogy. This balance is not only about the way cognition can control emotion, or the way emotion can wrest control of cognition, but the way in which the interactive and mutual feedback mechanisms between them can contribute to coping effectively with the challenges of problem solving.

A small change in the phrasing of a math problem can transform a routine question into an unexpected one, leading to an enjoyable activity that arouses curiosity. In this paper we present an instructional process that combines elements of inquiry and discovery, construction and dynamic simulation. We describe how it is possible to independently achieve development of mathematical knowledge and discovery of relationships among mathematical behaviors and features. We discuss the contribution of different exercises toward achieving pedagogical goals such as: arousing curiosity, development of research skills, generalization, and most of all, uncovering the beauty of mathematics. This article demonstrates a problem whose properties allow for integrating research in a dynamic environment with a mathematical experience. The original problem is characterized by the use of different proof approaches, and by inquiry into the consequences of a change taking place in the problem and its solution.

The understanding that description of the learning experience is a story, and that the story may be told in different ways, serves as the basis of this work.

Using tools and applications for the learning and pedagogical process within – and without – the curriculum is commendable, but does not suffice for a change in the nature of the learning process. Moving the stress from 'how' to 'why' calls out for investigatory processes at the stage of experimentation, discovery, hypothesis, generalization and reasoning. Problem solving must remain at the essence of learning math, but the problems are meant to be phrased as problems of state analysis, considerations of possible methods of solution and selection of an optimal solution. Let us stress that technology has much potential to offer efficient and diverse means of stimulating experiential learning and to lead to improved learning, and we will make use of it later on, but this is not a necessary condition; we will bring an example of this as well.

We begin with an exciting example[[1]](#footnote-1) that was able to surprise veteran teachers as well as us; later on we will present additional problems in the same vein. What the examples share is not their content but the way they are presented. Instead of presenting each exercise as a technical assignment, just one from a long list, each example may be opened up into a research activity to be performed independently, with a team of teachers, or with the class.

So, what is it that could surprise us about a cubic polynomial? We encounter these functions in the classroom when we study real analysis. Other than calculating where the function crosses the axes (formulas which may be somewhat complex), the exercises shown are classic – finding tangent equations, extrema and intervals where the function increases/decreases.

Before beginning, for the sake of the surprise, we recommend the following warm-up exercise:

* Sketch **by hand** on tracing **paper,** a cubic function with two extrema.
* Draw a horizontal line – for simplicity it can be the -axis – that intersects the function at three points.
* Mark the midpoint on the graph between the two leftmost points [that is, if the values are then mark the graph at the point ].
* Now draw a tangent to the function at that point.

We now repeat the process using GeoGebra or desmos.
We encourage you to try it on your own; here's a [link](https://www.geogebra.org/m/HWMRDXe9) to the app where we see a scrollbar with the steps of the construction.

* Step 1 – three points are selected on the axis:
* Step 2 – the function is drawn, passing through all three points:
 (the coefficient before the parentheses is there to make the graph 'nicer'; that is, less steep).
* Step 3 – define the point
* Step 4 – we draw a tangent to the parabola through that point [alternatively, in GeoGebra there is a tool for drawing tangents, and in desmos as well; we will enter ].

So we've drawn a tangent. But wait a minute – where does that tangent meet the function again?

For those of you who haven't yet opened up an app – now's the time to do so! Don't worry, we'll show more sketches later on, but we urge you try it out in the dynamic environment; here's the [link to the prepared app](https://www.geogebra.org/m/HWMRDXe9) once again.

What's going on here?

Many questions crop up here:

1. Does this always happen?
2. Does need to hold true?
3. Must points be distinct?
4. What's so special about the midpoint?
5. Why didn't the results look like that in the hand drawing? (At least in our first sketch the results were much less impressive)
6. Can we generalize from this case to higher order polynomials?

Using the app, we could get a gut feeling about the first three points raised here. Yes, it always happens, order doesn't matter, and the points may even converge!

Spoiler alert – if you want to find the proof yourself, put the paper down, go back to the drawing board, formulate your assertion, and after you have a proof (or a direction toward a proof), continue reading.
Let us begin with two sketches. In the first, holds true. In addition to the midpoint and its tangent, there is also a line through the origin with the same slope as the tangent.



Looking at the next sketch, this time holds true, but once again there is a line through the origin parallel to the tangent of the midpoint. Notice that both sketches have the same values of ; only has changed. Do you see a relationship between the slopes of the tangents in the two drawings?



Now let's move on to formulating the observed behavior.

Assertion 1:

Look at the function:

The tangent to the graph of the function at the point satisfies the following:

1. Its slope depends only on and does not depend on the value of .
2. It passes through the point on the graph.

Theoretically but the assertions and proofs do not rely on real properties and hold true for every field (with the proper indicators). Additionally, note that there is no assumption in the proof on the ordering of or that they are necessarily distinct.

The app does a very good job at convincing us of the truth of the assertion, yet it is not a proof. In the classroom we can add the following sentence after an interesting observation before showing the proof – "It's worthwhile proving, isn't it?" [[2]](#footnote-2)

Proof:

The function is defined as the product of three factors, but in order to prove the assertion we'll look at is as a product of two factors – the first is , and the second is . Let us now take the derivative using the product rule of differentiation:

if we denote , then is a quadratic function and its vertex is at the point ; therefore
and so

Thus the slope of the tangent to the graph of the function at the point is .

In other words, the slope is dependent upon and not on the value of . We've thus proven Part A of the assertion.

If it happens to be that then the point and converge, and the assertion holds as a trivial case.

Otherwise, let us calculate the slope of the line that passes through the points and :

Since
it follows that

There is only one line that passes through the point with the slope ; thus the point is on the tangent.

Does this assertion apply to cubic polynomials only? Can we generalize to higher order polynomials, or even to other functions?

Visual observation often brings us to generalizations or deeper insights of a problem. In this case it is the structure of the proof that calls out for a simple generalization, and also sheds some light on what is so special about the point .
The following assertion generalizes Assertion 1 and in a similarly manner shows a more general case. Here's a [link](https://ggbm.at/WQNGuPQF) to the app that demonstrates the assertion. You may play with the markings in the app and change the value as well as of the given function .

Assertion 2

Let us assume that is some function, not necessarily a standard parabola, and any .
We also define a function .

The following holds:

1. If we assume that is a differentiable function, then at the point for which , it holds that the tangent to the function at the point is not dependent on the value of and is equal to .
2. Given the point such that , it holds that the tangent of the line that passes through two points does not depend on the value of , and is equal to .

Proof

1. The calculation here is straightforward:
2. Here, too, the proof is immediate – the slope of the line equals:
.

We may immediately conclude from these two remarks that for every point for which , the statement that the point is on the tangent to the function at point holds true.

That is, Assertion 1 is an instance of Assertion 2, as it is a case where the function is a quadratic function of the form .The point is the only point in which holds, and thus it follows from Part A that the slope of the tangent to the function that passes through it does not depend on the value of – which is the essence of Part A of Assertion 1.But our big surprise – that the tangent through the point passes through the point on the graph where – actually follows from Part B of Assertion 2, since either the points converge, or they are unique, and in that case only one line could pass through the point with slope .

We could certainly stop here, but perhaps we could gain some more insight into our ability as teachers to introduce content in the classroom. In this article we wish to adjust this procedure to the particular class in terms of time and suitable level.

Our claim is that it is not necessary to wait for striking results – that is, for unusual cases – that are beyond the syllabus in order to engage students in inquisitive study in the classroom[[3]](#footnote-3).

In this article we would like to breathe life into existing exercises as we saw in the cases above. An ordinary math syllabus lends us plenty of such opportunities. Not every lesson – and not every exercise – must be presented in this way, but there is no reason not to do so with some of the content. Exercises that have a component or facet of a dynamic visual nature call out for this kind of presentation, especially with the availability of tools like desmos and GeoGebra.

Four textbook examples of real analysis and geometry at different levels are presented below, each allowing for specification of a particular mathematical condition from another perspective.

One of our examples lacks the dynamic visual component.

We begin with geometry. Geometry exercises call out for presentation in the manner described.

Most textbook exercises have an accompanying figure and appear in the format – Given:... Prove:...
If we delete the figure and what needs to be proven and remain with the givens only, we have fertile ground for research.
Yes, it demands much more preparation, and not every exercise can be performed this way, nor every lesson (sometimes we simply want them to carry out a list of exercises, and that, too, has value).

Example (1) – geometry:

When an exercise appears as a figure with 'givens' and what needs to be proven, the element of construction and discovery are completely absent. Below is an adapted example based on a standard exercise from "Mathematics – Five Units B1" [[4]](#footnote-4).

You are given [the following app](https://www.geogebra.org/m/XYamkdpP). In the app itself all you have is a circle and a scrollbar. Every stage in the scrollbar adds another 'given'. Instead of telling the class what the 'given' is, ask them to describe it.

In the final stage, two angles are indicated. Instead of drawing attention to the fact that one angle is twice the other – you may ask the class which angle is greater. And now – let's move the point .

And the question is – when is angle greater than angle ?

In practice this will never occur, and it is safe to assume that while studying the figure one of the students will notice that perhaps the larger angle is not only greater than the other, but it is exactly twice as large.
Discovery! Hypothesis! And what remains is – the proof.

Example (2) – geometry:

And what happens if there's no computer available, or we didn't have a chance to prepare or find the right app?

The method of presenting the exercise below has been inspired by "Kol Hakita, Kol Hazman" [The Entire Class, all the Time] by Kobi Guterman (Guterman, 2008). We ask **all students to please** sketch a circle in their notebooks with a diameter whose endpoints are . Do not proceed until everyone has drawn their figure. Next, select points each in a different half of the circle (connecting them by a chord), and some point on the diameter[[5]](#footnote-5). From the point drop perpendiculars to the chords , which meet the chords at the points respectively.
When everyone (or at least most) have completed, ask whether anyone has a hypothesis about the relationship between the chord and the line segment .

Included are several sketches. (You may have even drawn one in the margin yourself). Do you as readers have a hypothesis?





Someone dares to say that in their drawing they look parallel. Now you may ask the class – did it happen to come out parallel in anyone else's drawing? Is there anyone in whose drawing it did *not* come out parallel?[[6]](#footnote-6) Again, the element of surprise leads to discovery by raising a hypothesis, formulating an assertion, and proving it.

Now let's move on to an example in real analysis. Here, too, the dynamic and visual components are prominent, and so they call out for this method for the topic or for a single exercise. Two extreme value problems are described below.

The introduction to the book "Lilmod Ulilamed Analyza (Learning and Teaching Analysis)" (Ministry of Education, 2013) states that where possible, it is advisable to teach mathematics by combining a variety of pedagogical approaches, and that learning – including teachers' learning – is more meaningful when it is an active learning process.

Like the activities in the book, we present 'regular exercises' through which teachers could further develop their own mathematical knowledge, while at the same time experience the learning potential encapsulated in these activities.

Some of these activities may be taken and used in the classroom as is, while others require some adjustment in order to suit them to the particular class.

These exercises are designed to create new vantage points, to sharpen conceptual understanding, to see connections among different mathematical topics and principles and to demonstrate different applications of analysis for diverse student levels and populations.

Getting used to thinking about what happens with a small change in the given data. Such ideas should certainly be performed together with the students by composing the right questions.

Example (3) – extreme value problems:

The interesting case of a tangent to a cubic polynomial as well as the two other examples cited may leave a mistaken impression that creating such an experience only works with top level students or with advanced material. Here we present a standard extreme value problem, a particularly simple one at that. One of the first extreme value problems that appear in any textbook would be something like the following (the example is taken from Benny Goren's "Mathematics for 4-5 Units, Book A", p. 775, exercise 1).
This problem is likely to be found in any class learning real analysis.

 A rectangle is bounded between the graph of the line and the axes.
Vertex P of the rectangle is on the line. Find the rectangle of maximum area.

Since this is one of the first extreme value problems, you may want to begin by asking questions relevant to the students' experience with these kinds of exercises so far, such as:

* What is the area of the rectangle when ?
* For what point will the resultant rectangle have an area of 10?

Using [the app shown here](https://www.geogebra.org/m/En66u6g2) the students may find the answers immediately without performing calculations. (Of course you may also describe the calculations to be performed, but that's not the point of this exercise.) You may want to ask what would happen if the point would be on one of the axes, and in general what value would the area of a rectangle created in this way approach. The question that should follow from this discussion is – when does the area reach a maximum?
Of course this needs to be solved analytically as well afterwards (by differentiation or by using properties of the parabola).
What's interesting in this exercise is that it is so simple – by its very presentation a visual experience is create in which mathematical behavior is observed, formulated, and proven. In one particular class where we taught this, it was hypothesized that the point of maximum must necessarily be the midpoint between the two intersections of the line with the axes.
Immediate visual examination of this hypothesis was possible on the classroom computer[[7]](#footnote-7), just like [the app shown here](https://www.geogebra.org/m/Br94D3MM) [of course this app looks much better than the one we improvised in class]. Then we proved it analytically, too. The proof is really quite straightforward, and here in our first lesson of extreme value problems we were already engaging in some degree of research.

Example (4) – extreme value problems:

Later on in the chapter the following problem appears:

A rectangle is bounded between the graph of the parabola and the -axis in the first and second quadrants, such that the edge lies on the -axis. What must the length of the edge be for the area of the rectangle to be maximal?

We recommend leaving the book closed and presenting the problem slightly differently using [this app](https://www.geogebra.org/m/ZgbSttzV). While we're on the subject of extreme value problems, which are of course completely ordinary and not particularly interesting in themselves, we suggest bringing up a common rectangle-related misconception – as the area of the rectangle increases, so does the perimeter, and conversely. In addition, this is an opportunity to look at two rectangles where the second is half the first, thus we expect that their extrema (of the area and the perimeter) would be at the same point for both rectangles.

Instead of constructing only the rectangle specified in the original problem, let us construct two rectangles – one as indicated in the original problem, and the second bounded by the graph of the parabola and the axes in the first quadrant only. Ask questions about the perimeter and area of the two rectangles.
A. What happens at the boundaries?
B. Based on the values we observe visually, and the values we saw at the boundaries, do we expect to be looking for minimum area/perimeter or maximum?
C. Where do we in fact obtain the extremum? (use the app only; no calculations)

Note: when the students first encounter the problem, they already know that they are looking for extrema, but they don't know whether they're minima or maxima. They students are meant to **make that call by observation.**
After completing their answers to the extreme value problem, which is the central topic under study at this point, you may proceed to ask questions about the behavior of the function.
 D. Are the points of maximum area and maximum perimeter the same?

E. Are the maxima attained at the same point for the two rectangles (in one yes, and in the other not)? What's the difference between them?

The benefit here is twofold – we've experienced discovery in an ordinary extreme value problem, and we've addressed a common misconception in the relationship between area and perimeter of a rectangle.
We would like to add that in some classes the discussion could move ahead as follows: In this case, as in the previous example, can we find the maximum point when the original function is defined parametrically and not as a fixed function?

**Discussion**

Including emotional considerations in math education and the desire to give students an experiential feeling stems from an understanding that cognition and emotion should not be artificially separated. Math education is not a purely cognitive field to be contrasted with life-skill lessons which focus purely on emotion. The stories of learning that students create for themselves and those created by the teachers for the students have a large impact on their perception as learners, on their achievements, and on their successes and failures.

Many students report low motivation for learning mathematics. One method proposed by this article is to create a mathematical experience as a fundamental component for the learner's progress. When we say a 'mathematical experience' we are referring to the emotional variables related to mathematics that often serve as agents to success or failure. One way to bring students to be fond of mathematics independent of their level of knowledge, is to transform the learning process to an active one of giving meaning and creating knowledge, processing that knowledge and changing it.

In this article we propose that learners improve their skills and expand their knowledge base. Some of the examples allowed discovery and research by using dynamic tools. Use of visualization is essential in order to find a way to bring students to understanding proofs by reading them or by listening to their presentation, to develop their ability to form their own proofs and to write them up in an orderly and convincing manner by convention. No doubt that activities of this nature enable much deeper thinking and provide students with elements of surprise and interest that do not exist in standard tasks. We would like to stress, however, that this is not a necessary condition. In the first example (the tangent to a cubic polynomial), the first behavior of the function was in fact "discovered" with the help of an app, and there's nothing more convincing than what our eyes show us, but the discovery of the more general case came about during the formal proof. The second example introduced a way to engage the class, with the profusion of the various static sketches also bringing with it a classroom dynamic and drawing in all the students at all levels.

In his book "Mind in Society", Vygotsky introduces the "zone of proximal development"; that is, the distance between the level of development of the learner, identified with what he could do on his own, and his proximal level of development, implying what he could do when assisted by a peer with a higher skill set. This concept brought him much popularity since it emphasizes the role of the teacher as responsible for planning and timing the learning and finding appropriate pedagogical methods (Arnon, 1994). We, as teachers, therefore, must see ourselves as peer and partner to the learner as well as the ones with expertise who also function as role models. A student needs a teacher in order to appropriate technological and mathematical mechanisms and in order to make use of them for developing thinking during the problem-solving process. It is the story that a student tell himself while grappling with a mathematical problem that is the important one (Dunlap & Lowenthal, 2013).

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1. Thank you to Dr. Sabine Segre who referred us to the material from Texas Instruments [↑](#footnote-ref-1)
2. Thanks to Tzipi Eyal for the phrase "It's worthwhile proving, isn't it?" [↑](#footnote-ref-2)
3. You may want to read an instructional process about the example we opened the article with, performed with a group of students in a calculus course (Nabb, 2013) [↑](#footnote-ref-3)
4. Benny Goren, page 776 exercise 22. [↑](#footnote-ref-4)
5. Most of the time there will be at least one student in the class who will ask if it's important where the point $E$ lies relative to the chord $CD$, and this is an excellent opportunity to suggest that part of the class select the point $E$ between $A$ and the chord, while the other part of the class between $B$ and the chord, and even have one student place it on the extension of the diameter. [↑](#footnote-ref-5)
6. Usually there is such a student; you may ask them the check whether their sketch is accurate. [↑](#footnote-ref-6)
7. This activity was performed in Esther Gruenhut's classroom When describing the activity, she said, "I admit that I first thought that it's particular to the fact that we had a straight line with a slope of $2$" [↑](#footnote-ref-7)