

An Unbiased Estimator of the Causal Effect on the Variance based on the Back-door Criterion in Gaussian Linear Structural Equation Models

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Abstract

This paper assumes a context in which cause-effect relationships between random variables can be represented by a Gaussian linear structural equation model and the corresponding directed acyclic graph. We consider the situation where we observe a set of random variables satisfying the so-called back-door criterion. When the ordinary least squares method is utilized to estimate the total effect, we formulate the unbiased estimator of the causal effect (the estimated causal effect) on the variance of the outcome variable with external intervention in which a treatment variable is set to a specified constant value. In addition, we provide the variance formula for the estimated causal effect on the variance. The variance formula proposed in this paper is exact, in contrast to those in most previous studies on estimating causal effects.

Keywords: Causal effect, Identification, Path diagram, Structural causal model, Total effect
2020 MSC: Primary 62D20, Secondary 62H22

1. Introduction

1.1. Background

Statistical causal inference using linear structural equation models (linear SEMs) has been widely used to clarify cause-effect relationships between random variables in fields such as sociology, economics, and biology, and its origin can be traced back to path analysis (Wright, 1923, 1934). Statistical causal inference has been redeveloped as the theory of structural causal models (Pearl, 2009).

When a linear SEM is given as a statistical model to describe cause-effect relationships between random variables, the important aspects are direct, indirect, and total effects (Bollen, 1989). According to Bollen (1987, p. 40), the direct effect can be intuitively defined as “those influences unmediated by any other variable in the model” and the indirect effect as “those influences mediated by at least one intervening variable.” Here, an “intervening variable” is a random variable that could be affected by a treatment variable and in turn have an effect on an outcome variable. The total effect is defined as the sum of direct and indirect effects. In the framework of statistical causal inference using linear SEMs, the total effect also means the amount of the change in the expected value of an outcome variable when a treatment variable is changed by one unit due to external intervention. In this article, “identifiable” indicates that the total effect can be uniquely determined based on the variance-covariance parameters of observed variables. Causal understanding regarding differences in total, direct, and indirect effects contributes to evaluating how much of the causal effect of a treatment variable on an outcome variable is captured/not captured by intervening variables. The statistical method for promoting such causal understanding is called mediation analysis, and has its roots in the literature of linear SEMs, going back to path analysis (Wright, 1923, 1934) and continuing in the social sciences through the work of Duncan (1975), Baron and Kenny (1986), and Bollen (1989).

To evaluate the total effect, on which this paper focuses, statistical researchers in the field of linear SEMs have provided various identification conditions and estimation methods (e.g., Brito, 2004; Chan and Kuroki, 2010; Chen,

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2017; Henckel et al., 2019; Kuroki and Pearl, 2014; Maathuis and Colombo, 2015; Nandy et al., 2017; Pearl, 2009; Perković, 2018; Tian, 2004).

In characterizing the distributional change introduced by external intervention based on linear SEMs, there is no reason to limit our causal understanding to the change in the expected value of an outcome variable. In fact, Hernán and Robins (2022, p. 7) stated that

the average causal effect, defined by a contrast of means of counterfactual outcomes, is the most commonly used population causal effect. However, a population causal effect may also be defined as a contrast of functionals, including medians, variances, hazards, or cdfs of counterfactual outcomes. In general, a population causal effect can be defined as a contrast of any function of the marginal distributions of counterfactual outcomes under different actions or treatment values. For example, the population causal effect on the variance is defined as $var(Y^{a=1}) - var(Y^{a=0})$.

Actually, in practical science it is important to estimate the change in the expected value of an outcome variable due to external intervention (the causal effect on the mean). However, it is often also necessary to evaluate the variation (variance) of the outcome variable due to external intervention (the causal effect on the variance). For example, in the field of quality control, in order to suppress a defective rate of products effectively it is necessary to bring the outcome variable closer to the target value due to external intervention, thereby reducing the variation (minimizing the variance) of the outcome variable as much as possible. In this context, Kuroki (2008, 2012) and Kuroki and Miyakawa (1999a, b) discussed what happens to the variance of the outcome variable when conducting the external intervention. In medicine, according to Gische et al. (2021), the physician’s goal when treating hyperglycemia is to maintain the patient’s level of blood glucose within the euglycemic range (acceptable range) after the treatment (external intervention). The variance of the outcome variable by the external intervention, together with the physician’s knowledge, plays an important role in constructing the acceptable range and thus detecting a threat to the patient’s health.

Regarding the estimation accuracy (or, the variance) of the causal effect on the variance, Kuroki and Miyakawa (2003) discussed how, when the ordinary least squares method is utilized to estimate the total effect, the asymptotic variance of the consistent estimator of the causal effect on the variance differs with different sets of random variables that satisfy the so-called back-door criterion (Pearl, 2009). In addition, Shan and Guo (2010) studied the results of Kuroki and Miyakawa (2003) from the perspective of a particular type of external intervention using more than one treatment variable. Shan and Guo (2012) also extended the variable selection criteria provided by Kuroki and Miyakawa (2003) from a deterministic to a stochastic intervention. Kuroki and Nanmo (2020) applied the results of Kuroki and Miyakawa (2003) to predict future values of the outcome variable when conducting an external intervention. Here, note that the existing estimators of the causal effect on the variance are consistent but not unbiased estimators. Estimation accuracy problems are key issues for statistical causal inference, and thus it is important to formulate the unbiased estimator of the causal effect on the variance with the exact variance. This is because the reliable evaluation of estimation accuracy of the causal effect on variance plays an important role in the success of statistical data analysis, which aims to evaluate what would happen to the outcome variable when conducting external intervention based on non-experimental data.

This paper assumes that cause-effect relationships between random variables can be represented by a Gaussian linear SEM and the corresponding directed acyclic graph. In the situation where we observe a set of random variables satisfying the back-door criterion, when the ordinary least squares method is utilized to estimate the total effect, we formulate the unbiased estimator of the causal effect on the variance—namely the unbiased estimator of the variance of the outcome variable with external intervention in which a treatment variable is set to a specified constant value. In addition, we provide a variance formula for the unbiased estimator of the causal effect on the variance. The variance formula proposed in this paper is exact, in contrast to those in most previous studies on estimating causal effects.

1.2. Motivating Example

To motivate our problem, consider a case study reported by Okuno et al. (1986) of setting up coating conditions for car bodies. According to Okuno et al. (1986), since car bodies are coated both to increase rust protection and to enhance the car’s visual appearance, a certain level of coating thickness must be ensured in the coating process. At that time, the coating process was conducted by human operators who sprayed the car bodies with the paint. This was dependent on operators’ skills and might cause low transfer efficiency. Okuno et al. (1986) collected non-experimental

Table 1: Sample correlation matrix (Okuno et al., 1986)

| | X_1 | X_2 | X_3 | X_4 | X_5 | X_6 | X_7 | X_8 | X_9 | X_{10} | Y |
|----------|--------|--------|--------|--------|--------|--------|--------|--------|--------|----------|--------|
| X_1 | 1.000 | -0.678 | -0.215 | 0.230 | 0.040 | 0.116 | 0.338 | 0.002 | 0.145 | -0.496 | -0.198 |
| X_2 | -0.678 | 1.000 | 0.241 | -0.442 | -0.024 | 0.005 | -0.422 | -0.590 | -0.509 | 0.684 | 0.463 |
| X_3 | -0.215 | 0.241 | 1.000 | -0.201 | 0.004 | -0.067 | 0.208 | -0.007 | -0.082 | 0.307 | 0.292 |
| X_4 | 0.230 | -0.442 | -0.201 | 1.000 | 0.191 | -0.286 | 0.287 | 0.446 | 0.521 | -0.477 | -0.614 |
| X_5 | 0.040 | -0.024 | 0.004 | 0.191 | 1.000 | 0.291 | 0.117 | 0.034 | -0.048 | 0.010 | -0.151 |
| X_6 | 0.116 | 0.005 | -0.067 | -0.286 | 0.291 | 1.000 | 0.057 | -0.123 | -0.147 | 0.178 | -0.226 |
| X_7 | 0.338 | -0.422 | 0.208 | 0.287 | 0.117 | 0.057 | 1.000 | 0.251 | 0.287 | -0.122 | -0.113 |
| X_8 | 0.002 | -0.590 | -0.007 | 0.446 | 0.034 | -0.123 | 0.251 | 1.000 | 0.761 | -0.342 | -0.551 |
| X_9 | 0.145 | -0.509 | -0.082 | 0.521 | -0.048 | -0.147 | 0.287 | 0.761 | 1.000 | -0.571 | -0.431 |
| X_{10} | -0.496 | 0.684 | 0.307 | -0.477 | 0.010 | 0.178 | -0.122 | -0.342 | -0.571 | 1.000 | 0.282 |
| Y | -0.198 | 0.463 | 0.292 | -0.614 | -0.151 | -0.226 | -0.113 | -0.551 | -0.431 | 0.282 | 1.000 |

data on the coating process in order to examine the process conditions and increase the transfer efficiency; these data were important for creating a stable automated manufacturing process. The sample size in this study is 38 and the observed variables of interest are the following:

Coating conditions: dilution ratio (X_1), degree of viscosity (X_2), temperature of the paints (X_8)

Spraying conditions: gun speed (X_3), spray distance (X_4), air pressure (X_5), pattern width (X_6), fluid output (X_7)

Environmental conditions: air temperature (X_9), degree of moisture (X_{10})

Response: transfer efficiency (Y), defined as “the coated paint volume/consumption of paint” $\times 100\%$

According to Okuno et al. (1986), dilution ratio (X_1) and spray distance (X_4) are easily controlled. Degree of viscosity (X_2), gun speed (X_3), air pressure (X_5), and pattern width (X_6) can be controlled to some extent. Fluid output (X_7) and temperature of the paints (X_8) result from other factors and are difficult to control. Finally, air temperature (X_9) and degree of moisture (X_{10}) are environmental conditions that cannot be controlled. In addition, Okuno et al. (1986) also considered “wind speed” (environmental condition), “solid content” (coating condition), and other variables as factors that might have an effect on the transfer efficiency (Y). However, these factors were not observed since, according to Okuno et al. (1986), it seemed sufficient to observe the ten variables already listed to achieve their aim.

Concerning the coating process, Okuno et al. (1986) provided the sample correlation matrix shown in Table 3. By applying conventional stepwise regression analysis to Table 3, they obtained the following regression model:

$$Y = -0.636x_4 - 0.465x_6 + 0.189x_7 - 0.372x_8. \quad (1)$$

From the regression model (1) the transfer efficiency (Y) can be increased by controlling X_4 , X_6 , X_7 , and X_8 , but note that both the fluid output (X_7) and the temperature of the paints (X_8) are difficult to control in practice. In addition, in order to establish a stable coating process it is important to understand how the variation of the transfer efficiency (Y) would be changed by external intervention, because an increase in the variation of the transfer efficiency (Y) could lead to development of an unstable process. However, it is difficult to understand from equation (1) how the variation in the transfer efficiency (Y) would change under external intervention: the analysis should not simply be based on statistical aspects, rather it is desirable to describe the cause-effect relationships as a directed graph (called a causal path diagram) derived from the analyst’s causal knowledge. Combining this causal knowledge with statistical data, statistical causal inference using linear SEMs enables us to evaluate the variation of the transfer efficiency (Y) due to external intervention through (non-experimental) statistical data collected on the current coating process.

To present our results here, following Kuroki (2008, 2012) we assume that the cause-effect relationships in the coating process are given in Figure 1. For example, a directed edge from X_1 to X_2 ($X_1 \rightarrow X_2$) intuitively means that X_1 could cause X_2 directly, while a directed path from X_1 to X_7 without a directed edge from start to end point (i.e. here,

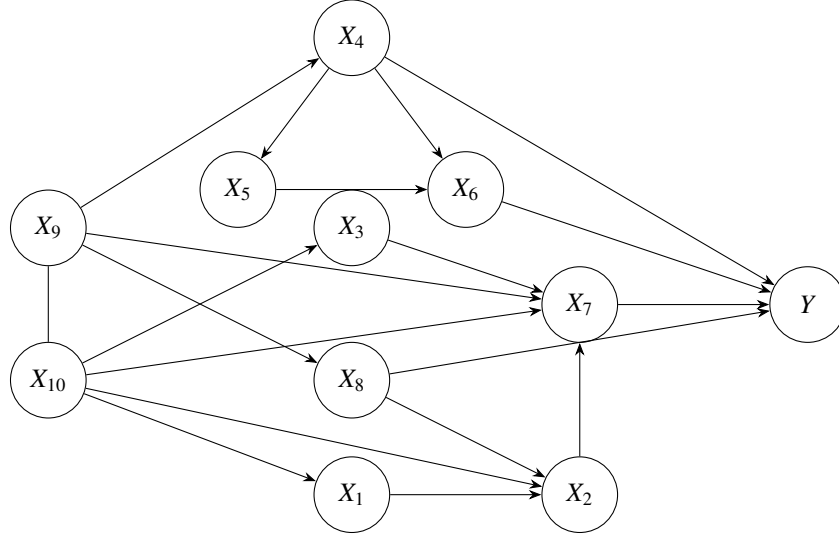


Figure 1: Causal path diagram for the coating process (Kuroki, 2012)

$X_1 \rightarrow X_2 \rightarrow X_7$) means that although X_1 cannot affect X_7 directly, the effect of X_1 on X_7 could be mediated by X_2 . This paper will not discuss the statistical inference problem posed by Figure 1. Refer to Kuroki (2012) for further details on this case study.

Here, under the assumption that $X_1, X_2, \dots, X_{10}, Y$ follows a multivariate normal distribution with zero mean vector and the variance-covariance matrix shown in Table 3, we evaluated unbiased estimators (17) and the consistent estimators (22) of the causal effect on the variance of the transfer efficiency (Y) 5000 times based on the sample size of 38. Table 2 reports the basic statistics of the unbiased estimators (17) and the consistent estimators (22) when a set of variables are utilized to identify the causal effects.

Regarding the causal effect on the variance, from the “Estimates” rows of Table 2 we see that although the consistent estimators are different from the values of equation (11), the unbiased estimators are close to the values of equation (11) even for small sample sizes ($n = 38$). In particular, regarding the external intervention on dilution ratio (X_1), while the unbiased estimators show that the external intervention could reduce the variation of transfer efficiency (Y), the consistent estimators imply that it does not do so. Such differences may require significant practical judgments: to establish a stable manufacturing process and increase the transfer efficiency, the external intervention should be conducted from the viewpoint of the unbiased estimators rather than the consistent ones.

2. Preliminaries

2.1. Graph Terminology

A directed graph is a pair $G = (V, E)$ where V is a finite set of vertices and $E \subset V \times V$ is a set of directed edges (\rightarrow) or ordered pairs of distinct vertices. If $(a, b) \in E$ for $a \neq b \in V$, then G contains the directed edge from vertex a to vertex b (denoted by $a \rightarrow b$); a is said to be the parent of b and b the child of a . Two vertices are adjacent if there is a directed edge between them. A path between a and b of length m is a sequence $a = a_0, a_1, \dots, b = a_m$ of distinct vertices such that a_i and a_{i+1} are adjacent for $i = 1, 2, \dots, m - 1$. A directed path from a to b of length m is a sequence $a = a_0, a_1, \dots, b = a_m$ of distinct vertices such that $a_i \rightarrow a_{i+1}$ for $i = 1, 2, \dots, m - 1$. If there is a directed path from a to b , then a is said to be an ancestor of b and b a descendant of a . In particular, $(a, b) \in E$ for $a, b \in V$ is not only a directed edge from a to b but also a directed path from a to b of length 1. In this case, a is both a parent and an ancestor of b , while b is both a child and a descendant of a .

Denoting the set of descendants of a as $de(a)$, the vertices in $V \setminus (de(a) \cup \{a\})$ are said to be the nondescendants of a . A vertex is said to be a collider if it is a common child of two or more other vertices; otherwise, it is said to be a non-collider. A directed path from a to b together with a directed edge from b to a forms a directed cycle. If a directed graph contains no directed cycles, then the graph is said to be a directed acyclic graph (DAG).

Table 2: Basis statistics of the coating process

| Treatment variable | Dilution Ratio (X_1) | | | | Spray Distance (X_4) | | | | | |
|--------------------|--------------------------|------------|-------------------|------------|--------------------------|------------|---------------------|------------|----------------|------------|
| Covariate | X_{10} | | $\{X_9, X_{10}\}$ | | X_9 | | $\{X_7, X_8, X_9\}$ | | $\{X_7, X_8\}$ | |
| Equation (11) | 0.975 | | 0.963 | | 0.629 | | 0.631 | | 0.640 | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.989 | 1.022 | 0.972 | 1.023 | 0.627 | 0.649 | 0.632 | 0.679 | 0.636 | 0.667 |
| Equation (17)/(22) | 0.055 | 0.052 | 0.052 | 0.049 | 0.023 | 0.021 | 0.023 | 0.022 | 0.024 | 0.023 |
| var | 0.056 | 0.060 | 0.053 | 0.058 | 0.022 | 0.023 | 0.023 | 0.026 | 0.023 | 0.024 |
| Skewness | 0.576 | 0.575 | 0.377 | 0.372 | 0.515 | 0.513 | 0.502 | 0.495 | 0.514 | 0.505 |
| Kurtosis | 3.722 | 3.724 | 3.034 | 3.038 | 3.453 | 3.450 | 3.429 | 3.411 | 3.535 | 3.513 |
| Minimum | 0.366 | 0.378 | 0.303 | 0.318 | 0.186 | 0.194 | 0.247 | 0.265 | 0.216 | 0.227 |
| 1st Quartile | 0.822 | 0.851 | 0.808 | 0.852 | 0.521 | 0.540 | 0.525 | 0.565 | 0.531 | 0.558 |
| Median | 0.969 | 1.001 | 0.954 | 1.004 | 0.614 | 0.634 | 0.623 | 0.669 | 0.623 | 0.655 |
| 3rd Quartile | 1.130 | 1.167 | 1.122 | 1.179 | 0.719 | 0.744 | 0.727 | 0.780 | 0.729 | 0.764 |
| Maimimum | 2.285 | 2.351 | 1.878 | 1.945 | 1.278 | 1.320 | 1.340 | 1.412 | 1.395 | 1.460 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50,000 estimated causal effects on the variance; Equation(11): the causal effect on the variance from equation (11) with Table 1; Equation(17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 1; Var: empirical variances from 50,000 estimated causal effects on the variance.

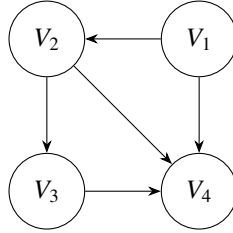


Figure 2: A simple causal path diagram

2.2. Linear Structural Equation Model

In this paper, it is assumed that cause-effect relationships between random variables can be represented by a Gaussian linear structural equation model (linear SEM) and the corresponding directed acyclic graph (DAG), called a causal path diagram and defined per Definition 1. Here, we refer to vertices in the DAG and random variables of the Gaussian linear SEM interchangeably.

Definition 1 (causal path diagram). Consider a DAG $G = (V, E)$ for which a set $V = \{V_1, V_2, \dots, V_p\}$ of p continuous random variables and a set E of directed edges are given. Then, G is a causal path diagram if the random variables are generated by a Gaussian linear SEM

$$V_i = \alpha_{v_i} + \sum_{V_j \in \text{pa}(V_i)} \alpha_{v_i v_j} V_j + \epsilon_{v_i}, \quad i = 1, 2, \dots, p, \quad (2)$$

satisfying the constraints entailed by G . Here, $\text{pa}(V_i)$ is a set of parents of $V_i \in V$ in G . In addition, letting $\mathbf{0}_p$ be a p -dimensional vector whose i th element is zero for $i = 1, 2, \dots, p$, $\epsilon_v = (\epsilon_{v_1}, \epsilon_{v_2}, \dots, \epsilon_{v_p})$ denotes a set of random variables that is assumed to follow a multivariate normal distribution with mean vector $\mathbf{0}_p$ and positive diagonal variance-covariance matrix Σ_{ϵ_v} . The constant parameters α_{v_i} and $\alpha_{v_i v_j}$ for $i, j = 1, 2, \dots, p$ ($i \neq j$) are referred to as the intercept of V_i and the causal path coefficient (or direct effect) of V_j on V_i , respectively. \square

As an example, consider the simple causal path diagram shown in Figure 2. We can see that (1) V_1 could be a direct cause of V_2 and V_4 , (2) V_2 could be a direct cause of V_3 and V_4 , and (3) V_3 could be a direct cause of V_4 . Then, the Gaussian linear SEM defined by Figure 2 is

$$V_1 = \alpha_{v_1} + \epsilon_{v_1}, \quad V_2 = \alpha_{v_2} + \alpha_{v_2 v_1} V_1 + \epsilon_{v_2}, \quad V_3 = \alpha_{v_3} + \alpha_{v_3 v_2} V_2 + \epsilon_{v_3}, \quad V_4 = \alpha_{v_4} + \alpha_{v_4 v_1} V_1 + \alpha_{v_4 v_2} V_2 + \alpha_{v_4 v_3} V_3 + \epsilon_{v_4} \quad (3)$$

where $\epsilon_{v_1}, \epsilon_{v_2}, \epsilon_{v_3}$, and ϵ_{v_4} independently follow a normal distribution with zero mean and non-zero variance.

The conditional independence induced by the Gaussian linear SEM (2) can be obtained from the causal path diagram G through d-separation (Pearl, 2009).

Definition 2 (d-separation). Let $\{X, Y\}$ and Z be disjoint sets of vertices in the DAG G . If Z blocks every path between the distinct vertices X and Y , then Z is said to d-separate X from Y in G . Here, the path p is said to be blocked by (a possibly empty) set Z if either of the following conditions is satisfied:

- (1) p contains at least one non-collider in Z ;
- (2) p contains at least one collider that is not in Z and has no descendant in Z . □

In Figure 2, both $\{V_2\}$ and $\{V_2, V_4\}$ satisfy Condition (1) of Definition 2 on the path $V_1 \rightarrow V_2 \rightarrow V_3$ since both sets include a non-collider V_2 . However, a collider (V_4) on the other paths is in $\{V_2, V_4\}$ but not in $\{V_2\}$. Thus, V_2 d-separates V_1 from V_3 but $\{V_2, V_4\}$ does not.

If Z d-separates X from Y in the causal path diagram G , then X is conditionally independent of Y given Z in the corresponding linear SEM (see e.g. Pearl, 2009). For example, in Figure 2, since $\{V_2\}$ d-separates $\{V_1\}$ from $\{V_3\}$, V_1 is conditionally independent of V_3 given V_2 .

2.3. Back-door Criterion

In this paper, for $X, Y \in V$ ($X \neq Y$), consider the external intervention in which X is set to be the constant value $X = x$ in the Gaussian linear SEM (2), denoted by $\text{do}(X = x)$. According to the structural causal models framework (Pearl, 2009), $\text{do}(X = x)$ indicates mathematically that the structural equation for X is replaced by $X = x$ in the Gaussian linear SEM (2).

Let $V = \{X, Y\} \cup W$ be the set of random variables in the causal path diagram G , where $\{X, Y\}$ and W are disjoint. Also let $f(x, y, w)$ and $f(x|\text{pa}(x))$ denote the joint probability distribution of $(X, Y, W) = (x, y, w)$ and the conditional probability distribution of $X = x$ given $\text{pa}(X) = \text{pa}(x)$, respectively. The causal effect of X on Y , denoted by $f(y|\text{do}(X = x))$, is defined as

$$f(y|\text{do}(X = x)) = \int_w \frac{f(x, y, w)}{f(x|\text{pa}(x))} dw \quad (4)$$

(Pearl, 2009). When equation (4) can be uniquely determined from the probability distribution of observed variables, the causal effect is said to be identifiable: that is, it can be estimated consistently. In this paper,

$$E(Y|\text{do}(X = x)) = \mu_{y|x} = \int_y y f(y|\text{do}(X = x)) dy, \quad \text{var}(Y|\text{do}(X = x)) = \sigma_{yy|x} = \int_y (y - \mu_{y|x})^2 f(y|\text{do}(X = x)) dy \quad (5)$$

give the causal effects of $\text{do}(X = x)$ on the mean and the variance of Y , respectively. $E(Y|\text{do}(X = x))$ and $\text{var}(Y|\text{do}(X = x))$ are also called the interventional mean and the interventional variance by Gische et al. (2021). In the Gaussian linear SEM (2), the first derivative of $E(Y|\text{do}(X = x))$ of Y , namely

$$\frac{dE(Y|\text{do}(X = x))}{dx} = \tau_{yx} \quad (6)$$

is called the total effect of X on Y . Graphically, the total effect τ_{yx} is interpreted as the total sum of the products of the causal path coefficients on the sequence of directed edges along all directed paths from X to Y . If the total effect τ_{yx} can be uniquely determined from the variance-covariance parameters of observed variables, then it is said to be identifiable and can be estimated consistently. The interpretation of the total effects in the Gaussian linear SEM (2) via path analysis (Wright, 1923, 1934) is also discussed in detail by Henckel et al. (2019) and Nandy et al. (2017).

Let $G_{\underline{X}}$ be the directed graph obtained by deleting all the directed edges emerging from X in the DAG G . The back-door criterion is a well-known identification condition of the causal effect (Pearl, 2009).

Definition 3 (back-door criterion). Let $\{X, Y\}$ and Z be disjoint subsets of V in the DAG G . If Z satisfies the following conditions relative to the ordered pair (X, Y) , then Z is said to satisfy the back-door criterion relative to (X, Y) :

1. no vertex in Z is a descendant of X ;
2. Z d-separates X from Y in $G_{\underline{X}}$. □

Other identification conditions of causal effects are known, such as the front-door criterion (Pearl, 2009) and the effect restoration (Kuroki and Pearl, 2014). However, this paper is only concerned with identification of a causal effect using the back-door criterion. As seen from Definition 3, the back-door criterion is not a statistical concept and cannot be tested through statistical data.

In Figure 2, both $\{V_2\}$ and $\{V_1, V_2\}$ satisfy the back-door criterion relative to (V_3, V_4) . However, $\{V_1\}$ does not satisfy the back-door criterion relative to (V_3, V_4) , since $\{V_1\}$ does not d-separate V_3 from V_4 in the graph G_{V_3} derived from Figure 2. For example, $\{V_1\}$ does not possess any non-collider (V_2) on the path $V_3 \leftarrow V_2 \rightarrow V_4$ and neither colliders nor their descendants lie on the path.

When \mathbf{Z} satisfies the back-door criterion relative to (X, Y) in the causal path diagram G , the causal effect of X on Y is identifiable and is given by

$$f(y|\text{do}(X = x)) = \int_{\mathbf{z}} f(y|x, \mathbf{z})f(\mathbf{z})d\mathbf{z} \quad (7)$$

(Pearl, 2009).

Here, we define some notation. For univariates X and Y and a set \mathbf{Z} of random variables, let μ_x and μ_y be the means of X and Y , respectively. In addition, let σ_{xy} , σ_{xx} , and σ_{yy} be the covariance between X and Y , the variance of X , and the variance of Y respectively. Using the prime notation ($'$) to represent the transpose of a vector or matrix, let Σ_{xz} , Σ_{yz} , and Σ_{zz} be the cross-covariance vector between X and \mathbf{Z} ($\Sigma_{zx} = \Sigma'_{xz}$), the cross-covariance vector between Y and \mathbf{Z} ($\Sigma_{zy} = \Sigma'_{yz}$), and the variance-covariance matrix of \mathbf{Z} , respectively. Now consider the regression model of Y on X and \mathbf{Z} ,

$$Y = \beta_{y \cdot xz} + \beta_{yx \cdot xz}X + B_{yz \cdot xz}\mathbf{Z} + \epsilon_{y \cdot xz}, \quad (8)$$

where $\epsilon_{y \cdot xz}$ is a normally distributed random variable with mean zero and variance $\sigma_{yy \cdot xz}$, while $\beta_{y \cdot xz}$, $\beta_{yx \cdot xz}$, and $B_{yz \cdot xz}$ are the regression intercept, the regression coefficient of X , and the regression coefficient vector of \mathbf{Z} in the regression model (8), respectively. According to the standard assumptions of linear regression analysis, $\epsilon_{y \cdot xz}$ is assumed to be independent of both X and \mathbf{Z} . Then, for a non-empty set \mathbf{Z} , letting

$$\sigma_{xy \cdot z} = \sigma_{xy} - \Sigma_{xz}\Sigma_{zz}^{-1}\Sigma_{zy}, \quad \sigma_{xx \cdot z} = \sigma_{xx} - \Sigma_{xz}\Sigma_{zz}^{-1}\Sigma_{zx}, \quad \Sigma_{zz \cdot x} = \Sigma_{zz} - \frac{\Sigma_{zx}\Sigma_{xz}}{\sigma_{xx}}, \quad \Sigma_{yz \cdot x} = \Sigma_{yz} - \frac{\sigma_{xy}}{\sigma_{xx}}\Sigma_{xz}, \quad \Sigma_{zy \cdot x} = \Sigma'_{yz \cdot x}, \quad (9)$$

the regression coefficient of X and the regression coefficient vector of \mathbf{Z} are given by $\beta_{yx \cdot xz} = \sigma_{xy \cdot z}/\sigma_{xx \cdot z}$ and $B_{yz \cdot xz} = \Sigma_{yz \cdot x}\Sigma_{zz \cdot x}^{-1}$, respectively, when $\sigma_{xx} \neq 0$, $\sigma_{xx \cdot z} \neq 0$, and both Σ_{zz} and $\Sigma_{zz \cdot x}$ are positive definite matrices.

When a set \mathbf{Z} of observed variables satisfies the back-door criterion relative to (X, Y) , then the total effect τ_{yx} is identifiable and is given by $\tau_{yx} = \beta_{yx \cdot xz}$ (Pearl, 2009). Consider the regression model (8) of Y on X and \mathbf{Z} . Then, letting

$$\sigma_{yy \cdot z} = \sigma_{yy} - \Sigma_{yz}\Sigma_{zz}^{-1}\Sigma_{zy} \text{ and } \sigma_{yy \cdot xz} = \sigma_{yy \cdot z} - \frac{\sigma_{xy \cdot z}^2}{\sigma_{xx \cdot z}}, \quad E(Y|\text{do}(X = x)) \text{ and } \text{var}(Y|\text{do}(X = x)) \text{ are formulated according to equation (7) as}$$

$$E(Y|\text{do}(X = x)) = \mu_{y|x} = \mu_y + \beta_{yx \cdot xz}(x - \mu_x) = \mu_y + \tau_{yx}(x - \mu_x) \quad (10)$$

and

$$\text{var}(Y|\text{do}(X = x)) = \sigma_{yy|x} = \sigma_{yy \cdot xz} + B_{yz \cdot xz}\Sigma_{zz}B'_{yz \cdot xz} \quad (11)$$

respectively (Kuroki and Miyakawa, 1999ab, 2003). Here, equation (11) shows that \mathbf{Z} behaves similarly to a random variable such as $\epsilon_{y \cdot xz}$ in equation (8) by conducting the external intervention $\text{do}(X = x)$; the external intervention may not reduce the variation of the outcome variable Y (Kuroki, 2012).

To conclude our discussion, we also consider the regression coefficient vector of \mathbf{Z} in the regression model of X on \mathbf{Z} :

$$X = \beta_{x \cdot z} + B_{xz \cdot z}\mathbf{Z} + \epsilon_{x \cdot z} \quad (12)$$

where $\epsilon_{x \cdot z}$ is a normally distributed random variable with mean zero and variance $\sigma_{xx \cdot z}$, while $\beta_{x \cdot z}$ and $B_{xz \cdot z}$ are the regression intercept and the regression coefficient vector of \mathbf{Z} , respectively. Here, $\epsilon_{x \cdot z}$ is also assumed to be independent of \mathbf{Z} . We denote the regression coefficient vector of \mathbf{Z} by $B_{xz \cdot z} = \Sigma_{xz}\Sigma_{zz}^{-1}$ when Σ_{zz} is a positive definite matrix.

3. Results

Let $\hat{\mu}_x$ and $\hat{\mu}_y$ be the sample means of X and Y , respectively. In addition, let s_{xx} , s_{yy} , s_{xy} , S_{zz} , S_{xz} , and S_{yz} be the sum of squares of X , the sum of squares of Y , the sum of cross-products between X and Y , the sum-of-squares matrix of Z , the sum-of-cross-products vector between X and Z ($S_{zx} = S'_{xz}$), and the sum-of-cross-products vector between Y and Z ($S_{zy} = S'_{yz}$), respectively. Then, for non-empty Z , letting

$$s_{xy.z} = s_{xy} - S_{xz}S_{zz}^{-1}S_{zy}, \quad s_{xx.z} = s_{xx} - S_{xz}S_{zz}^{-1}S_{zx}, \quad S_{zz.x} = S_{zz} - \frac{S_{zx}S_{xz}}{s_{xx}}, \quad S_{yz.x} = S_{yz} - \frac{s_{xy}}{s_{xx}}S_{xz}, \quad S_{zy.x} = S'_{yz.x}, \quad (13)$$

the unbiased estimators of $\beta_{yx.xz}$, $B_{xz.z}$, and $B_{yz.xz}$ of equations (8) and (12) are given through the ordinary least squares method by $\hat{\beta}_{yx.xz} = s_{xy.z}/s_{xx.z}$, $\hat{B}_{xz.z} = S_{xz}S_{zz}^{-1}$, and $\hat{B}_{yz.xz} = S_{yz.x}S_{zz.x}^{-1}$ respectively, when $s_{xx} \neq 0$, $s_{xx.z} \neq 0$, and both S_{zz} and $S_{zz.x}$ are positive definite matrices. Letting n and q be the sample size and the number of random variables respectively in Z , for $q < n - 2$,

$$\hat{\sigma}_{yy.xz} = \frac{s_{yy.xz}}{n - q - 2} = \frac{s_{yy.z} - \frac{s_{xy.z}^2}{s_{xx.z}}}{n - q - 2}, \quad \hat{\Sigma}_{zz} = \frac{1}{n - 1}S_{zz} \quad (14)$$

are also unbiased estimators of $\sigma_{yy.xz}$ and Σ_{zz} respectively, where $s_{yy.z} = s_{yy} - S_{yz}S_{zz}^{-1}S_{zy}$.

Under random sampling, when the total effect τ_{yx} is estimated as $\hat{\tau}_{yx} = \hat{\beta}_{yx.xz}$ through the ordinary least squares method in the regression model (8), the exact variance of $\hat{\beta}_{yx.xz}$ is given by

$$\text{var}(\hat{\beta}_{yx.xz}) = \frac{1}{n - q - 3} \frac{\sigma_{yy.xz}}{\sigma_{xx.z}} \quad (15)$$

for $q < n - 3$ (e.g., Kuroki and Cai, 2004).

The following theorem holds:

Theorem 1. Under the Gaussian linear SEM (2), suppose that Z satisfies the back-door criterion relative to (X, Y) in the causal path diagram G . When the ordinary least squares method is utilized to evaluate the statistical parameters in equations (10) and (11), the unbiased estimators of $\mu_{y|x} = E(Y|do(X = x))$ and $\sigma_{yy|x} = \text{var}(Y|do(X = x))$ are given by

$$\hat{\mu}_{y|x} = \hat{\mu}_y + \hat{\beta}_{yx.xz}(x - \hat{\mu}_x) \quad (16)$$

and

$$\hat{\sigma}_{yy|x} = \hat{\sigma}_{yy.xz} \left(1 - \frac{1}{n - 1} \left(q + \frac{\hat{B}_{xz.z}S_{zz}\hat{B}'_{xz.z}}{s_{xx.z}} \right) \right) + \hat{B}_{yz.xz}\hat{\Sigma}_{zz}\hat{B}'_{yz.xz} \quad (17)$$

respectively; $\hat{\mu}_{y|x}$ and $\hat{\sigma}_{yy|x}$ are the estimated causal effects of $do(X = x)$ on the mean of Y and the variance of Y , respectively. Further, for $q < n - 5$, the variances $\text{var}(\hat{\mu}_{y|x})$ of $\hat{\mu}_{y|x}$ and $\text{var}(\hat{\sigma}_{yy|x})$ of $\hat{\sigma}_{yy|x}$ are given by

$$\text{var}(\hat{\mu}_{y|x}) = \frac{1}{n} \left(\sigma_{yy.xz} + B_{yz.xz}\Sigma_{zz}B'_{yz.xz} \right) + \frac{\sigma_{yy.xz}}{(n - q - 3)\sigma_{xx.z}} \left((x - \mu_x)^2 + \frac{\sigma_{xx}}{n} \right) \quad (18)$$

and

$$\begin{aligned} \text{var}(\hat{\sigma}_{yy|x}) &= \frac{2(B_{yz.xz}\Sigma_{zz}B'_{yz.xz})^2}{n - 1} + \frac{2\sigma_{yy.xz}^2}{n - q - 2} \left(\left(1 - \frac{q}{n - 1} \right)^2 - 2 \left(1 - \frac{q}{n - 1} \right) \frac{q\sigma_{xx.z} + (n - 1)B_{xz.z}\Sigma_{zz}B'_{xz.z}}{(n - 1)(n - q - 3)\sigma_{xx.z}} \right. \\ &\quad \left. + E \left(\left(\frac{\hat{B}_{xz.z}S_{zz}\hat{B}'_{xz.z}}{(n - 1)s_{xx.z}} \right)^2 \right) \right) + \frac{2\sigma_{yy.xz}^2}{(n - 1)^2} \left(q + 2 \frac{q\sigma_{xx.z} + (n - 1)B_{xz.z}\Sigma_{zz}B'_{xz.z}}{(n - q - 3)\sigma_{xx.z}} + E \left(\left(\frac{\hat{B}_{xz.z}S_{zz}\hat{B}'_{xz.z}}{s_{xx.z}} \right)^2 \right) \right) \\ &\quad + \frac{4\sigma_{yy.xz}}{(n - 1)^2} \left((n - 1)B_{yz.xz}\Sigma_{zz}B'_{yz.xz} + E \left(\frac{(B_{yz.xz}S_{zz}\hat{B}'_{xz.z})^2}{s_{xx.z}} \right) \right) \end{aligned} \quad (19)$$

respectively, where

$$E\left(\left(\frac{\hat{B}_{xz,z}S_{zz}\hat{B}'_{xz,z}}{s_{xx,z}}\right)^2\right) = \frac{2q\sigma_{xx,z}^2 + 4(n-1)\sigma_{xx,z}B_{xz,z}\Sigma_{zz}B_{xz,z}}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} + \frac{2(n-1)(B_{xz,z}\Sigma_{zz}B_{xz,z})^2}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} + \frac{(q\sigma_{xx,z} + (n-1)B_{xz,z}\Sigma_{zz}B_{xz,z})^2}{(n-q-3)(n-q-5)\sigma_{xx,z}^2}, \quad (20)$$

$$E\left(\frac{(B_{yz,xz}S_{zz}\hat{B}'_{xz,z})^2}{s_{xx,z}}\right) = \frac{(n-1)(n(B_{yz,xz}\Sigma_{zz}B'_{xz,z})^2 + (B_{xz,z}\Sigma_{zz}B'_{xz,z})(B_{yz,xz}B'_{xz,z})^2 + \sigma_{xx,z}B_{yz,xz}\Sigma_{zz}B'_{yz,xz})}{(n-q-3)\sigma_{xx,z}}. \quad (21)$$

□

Both equations (16) and (18) are given by Kuroki and Nanmo (2020). The derivation of equations (17) and (19), which are the new results, is provided in the Appendix. Note that the assumption of Gaussian random variables in equation (2) is necessary to derive equation (19) but not to derive (17).

For a large sample size n , such that $n^{-1} \gg n^{-2} \approx 0$, the consistent estimator $\tilde{\sigma}_{yy|x}$ of $\sigma_{yy|x}$ can be given by

$$\tilde{\sigma}_{yy|x} = \hat{\sigma}_{yy,xz} + \hat{B}_{yz,xz}\hat{\Sigma}_{zz}\hat{B}'_{yz,xz}, \quad (22)$$

which shows that equation (22) yields a larger value than equation (17). The asymptotic variance of $\hat{\sigma}_{yy|x}$, $a.\text{var}(\hat{\sigma}_{yy|x})$, is given by

$$\begin{aligned} a.\text{var}(\hat{\sigma}_{yy|x}) &= \frac{2\sigma_{yy,xz}^2}{n} + \frac{2(B_{yz,xz}\Sigma_{zz}B_{yz,xz})^2}{n} + \frac{4\sigma_{yy,xz}}{n} \left(B_{yz,xz}\Sigma_{zz}B'_{yz,xz} + \frac{(B_{yz,xz}\Sigma_{zz}B'_{xz,z})^2}{\sigma_{xx,z}} \right) \\ &= \frac{2}{n} \left(\sigma_{yy,xz} + B_{yz,xz}\Sigma_{zz}B'_{yz,xz} \right)^2 + \frac{4\sigma_{yy,xz}}{n\sigma_{xx,z}} (B_{yz,xz}\Sigma_{zz}B'_{xz,z})^2. \end{aligned} \quad (23)$$

Letting $\beta_{yx,x} = \sigma_{xy}/\sigma_{xx}$, from $\beta_{yx,x} = \tau_{yx}$ and $B_{xz,z} = \Sigma_{xz}\Sigma_{zz}^{-1}$, calculating the covariance between X and the model for Y in equation (8) leads to the expression

$$\sigma_{xy} = \beta_{yx,xz}\sigma_{xx} + B_{yz,xz}\Sigma_{zz} = \tau_{yx}\sigma_{xx} + B_{yz,xz}\Sigma_{zz}, \quad (24)$$

which yields

$$B_{yz,xz}\Sigma_{zz}B'_{xz,z} = B_{yz,xz}\Sigma_{zz} = (\beta_{yx,x} - \tau_{yx})\sigma_{xx} \quad (25)$$

and

$$\sigma_{yy,xz} + B_{yz,xz}\Sigma_{zz}B'_{yz,xz} = \sigma_{yy,x} - B_{yz,xz}\Sigma_{zz}B'_{yz,xz} + B_{yz,xz}\Sigma_{zz}B'_{yz,xz} = \sigma_{yy,x} + \frac{(B_{yz,xz}\Sigma_{zz})^2}{\sigma_{xx}} = \sigma_{yy,x} + (\beta_{yx,x} - \tau_{yx})^2\sigma_{xx}. \quad (26)$$

From equation (26), the first term of equation (22), which is equivalent to equation (11), does not depend on the selection of the set \mathbf{Z} of random variables satisfying the back-door criterion (Kuroki, 2008, 2012). Likewise, from equation (25), $B_{yz,xz}\Sigma_{zz}B'_{xz,z}$ in the second term of equation (23) does not depend on the selection of \mathbf{Z} . Thus, the difference between selected sets of random variables depends on $\sigma_{yy,xz}/\sigma_{xx,z}$ in the second term of equation (22). Considering this and letting $\hat{\sigma}_{yy|x,z}$ be the estimated causal effect of $\text{do}(X = x)$ on the variance of Y to emphasize that \mathbf{Z} is utilized to estimate equation (11), the following theorem extends the variable selection criterion given by Kuroki and Miyakawa (2003) from the univariate case to the multivariate case.

Theorem 2. Under the Gaussian linear SEM (2), suppose that sets \mathbf{Z}_1 and \mathbf{Z}_2 of random variables satisfy the back-door criterion relative to (X, Y) in the causal path diagram G . When the ordinary least squares method is utilized to evaluate the statistical parameters in equations (10) and (11), if \mathbf{Z}_2 d-separates X from \mathbf{Z}_1 , then

$$a.\text{var}(\hat{\sigma}_{yy|x,z_1,z_2}) \leq a.\text{var}(\hat{\sigma}_{yy|x,z_2}) \quad (27)$$

holds, and if $\{X\} \cup \mathbf{Z}_1$ d-separates Y from \mathbf{Z}_2 , then

$$a.\text{var}(\hat{\sigma}_{yy|x,z_1}) \leq a.\text{var}(\hat{\sigma}_{yy|x,z_1,z_2}) \quad (28)$$

holds.

□

The proof of Theorem 2 follows immediately from the following lemma given by Kuroki and Cai (2004):

Lemma 1. *When $\{X, Y\} \cup \mathbf{Z}_1 \cup \mathbf{Z}_2$ follows a multivariate normal distribution, if X is conditionally independent of \mathbf{Z}_1 given \mathbf{Z}_2 , then*

$$\frac{\sigma_{yy \cdot xz_1 z_2}}{\sigma_{xx \cdot z_1 z_2}} \leq \frac{\sigma_{yy \cdot xz_2}}{\sigma_{xx \cdot z_2}} \quad (29)$$

holds, and if Y is conditionally independent of \mathbf{Z}_2 given $\{X\} \cup \mathbf{Z}_1$, then

$$\frac{\sigma_{yy \cdot xz_1}}{\sigma_{xx \cdot z_2}} \leq \frac{\sigma_{yy \cdot xz_1 z_2}}{\sigma_{xx \cdot z_1 z_2}} \quad (30)$$

holds. □

Intuitively, equation (27) shows that the estimation accuracy could be improved by adding \mathbf{Z}_1 , because \mathbf{Z}_1 is not correlated with X given \mathbf{Z}_2 and plays a role in decreasing the residual variance of Y . In contrast, equation (28) shows that the estimation accuracy could be worse, because adding \mathbf{Z}_2 may cause multicollinearity and thus increase the residual variance of Y . In Figure 1, since V_2 d-separates V_1 from V_3 , we know from Theorem 2 that

$$\text{a.var}(\hat{\sigma}_{v_4 v_4 | v_3, v_1 v_2}) \leq \text{a.var}(\hat{\sigma}_{v_4 v_4 | v_3, v_2}) \quad (31)$$

holds based only on the graph structure, without statistical data.

4. Numerical Experiments

This section will report numerical experiments conducted to examine statistical properties of the estimated causal effect on the variance for sample sizes $n = 10, 25, 50, 100, 500$, and 1000 . For simplicity, consider the DAG depicted in Figure 3 and the Gaussian linear SEM in the form of

$$Y = \alpha_{yx}X + \alpha_{yz_1}Z_1 + \epsilon_y, \quad X = \alpha_{xz_2}Z_2 + \epsilon_x, \quad Z_1 = \alpha_{z_1 z_2}Z_2 + \epsilon_{z_1}, \quad Z_2 = \epsilon_{z_2}, \quad (32)$$

where we assume the following two cases as the distribution (with mean zero) of $\epsilon_x, \epsilon_y, \epsilon_{z_1}$, and ϵ_{z_2} independently: (a) a normal distribution and (b) a uniform distribution. The matrices of the causal path coefficients of X, Y, Z_1 , and Z_2 shown in Table 3 are utilized for our purposes. In this situation, $\mathbf{Z} = \{Z_1\}, \{Z_2\}$, and $\{Z_1, Z_2\}$ satisfy the back-door criterion relative to (X, Y) . Cases 1 and 2 represent situations where the empty set also satisfies the back-door criterion relative to (X, Y) . Because X is independent of $\{Z_1, Z_2\}$ in Case 1, we obtain $\tau_{yx} = \beta_{yx \cdot x} = \beta_{yx \cdot xz}$ for \mathbf{Z} , and this information about \mathbf{Z} would asymptotically improve the estimation accuracy of the total effect τ_{yx} (Kuroki and Cai, 2004). In Case 2, because Y is conditionally independent of \mathbf{Z} given X , we also obtain $\tau_{yx} = \beta_{yx \cdot x} = \beta_{yx \cdot xz}$. However, this information about \mathbf{Z} does not asymptotically improve the estimation accuracy of the total effect τ_{yx} (Kuroki and Cai, 2004). Cases 3 and 4 represent situations in which \mathbf{Z} satisfies the back-door criterion relative to (X, Y) , but parametric cancellation occurs (Cox and Wermuth, 2014), where $\beta_{yx \cdot x} = 0$ and $\tau_{yx} = \beta_{yx \cdot xz} \neq 0$ hold in Case 3, whereas $\beta_{yx \cdot x} \neq 0$ and $\tau_{yx} = \beta_{yx \cdot xz} \simeq 0$ hold in Case 4. Case 5 represents an extreme situation in which the simple regression model of Y on X ,

$$E(Y|X = x) = \mu_y + \beta_{yx \cdot x}(x - \mu_x),$$

is orthogonal to the causal effect on the mean,

$$E(Y|\text{do}(X = x)) = \mu_y + \beta_{yx \cdot xz}(x - \mu_x);$$

that is, $\beta_{yx \cdot xz}\beta_{yx \cdot x} = \tau_{yx}\beta_{yx \cdot x} \simeq -1$ holds.

We simulated n random samples from a multivariate normal distribution of (X, Y, Z_1, Z_2) with a zero mean vector and the correlation matrices generated from each case of Table 3. For the causal effects on the variance, we evaluated both the unbiased estimator (17) and the consistent estimator (22) 50,000 times based on $n = 10, 25, 50, 100, 500$, and 1000 . Tables 4 and 5 report the basic statistics of equations (17) and (22) when $\{Z_1\}, \{Z_2\}$, and $\{Z_1, Z_2\}$ are utilized to identify the causal effects.

First, from the “Estimates” rows of Tables 4 and 5, in each case the consistent estimators are highly biased for the smaller sample sizes but become less biased for the larger sample sizes. In particular, the bias reduction speed based on the sample size depends on the correlation between X and Z : it seems that bias reduction is slower when X is highly correlated with Z . In contrast, the unbiased estimators are close to the true values even for the small sample sizes. However, as seen from the “Minimum” rows of Tables 4 and 5, the minimum values of the unbiased estimators when X is correlated with Z are negative at smaller sample sizes, but not for the larger sample sizes (the consistent estimators do not take negative values). In addition, from both the “Minimum” and “Maximum” rows of Tables 4 and 5, the sample ranges of the unbiased estimators when X is highly correlated with Z are wider than those of the consistent estimators in the smaller sample sizes. However, they become close to those of the consistent estimators in the larger sample sizes. Here, note that the sample ranges of the unbiased estimators are narrower than or close to those of the consistent estimators when X is uncorrelated with Z .

Second, from the “(17)/(22)” rows of Tables 4 and 5 (except for Case 2), equations (17) and (22) yield larger values for all sample sizes when Z_2 is selected than when either Z_1 or $[Z_1, Z_2]$ are selected. Also, equations (17) and (22) when $[Z_1, Z_2]$ is selected yield larger values than when Z_1 is selected. This implies that the relationships are consistent with the results obtained from Theorem 2. In contrast, in Case 2 with sample size $n \leq 25$, equation (17) yields a larger value when $[Z_1, Z_2]$ is selected than when Z_2 is selected, showing that in this case the relationships are different from the results obtained from Theorem 2. Thus, it appears that variation in estimation accuracy given the selected variables depends not only on the sample size but also on the multicollinearity between X and Z and the number of random variables included in Z : Theorem 2 holds for large sample sizes even when X is highly correlated with Z .

Third, comparing the empirical variances with the variance formula, equation (17) is relatively close to the empirical variances of the unbiased estimator for any sample size when ϵ_x , ϵ_y , ϵ_{z_1} , and ϵ_{z_2} follow the normal distribution. In contrast, when ϵ_x , ϵ_y , ϵ_{z_1} , and ϵ_{z_2} follow the uniform distribution, the differences between equation (17) and the empirical variances of the unbiased estimator are more significant for the smaller sample sizes, although not for the larger ones. In addition, when X is correlated with Z , the asymptotic variance (22) is not close to the empirical variances of the consistent estimator for the small sample sizes in each case. In particular, the differences between the asymptotic variance (22) and the empirical variances of the consistent estimator are significant when X is correlated with Z . However, the differences between the variables becomes smaller as the sample size increases.

Finally, it appears that in each case both unbiased and consistent estimators are highly skewed and heavy-tailed for small sample sizes, but slowly converge to normal distributions as the sample size increases. In particular, when X is correlated with Z , both unbiased and consistent estimators take large values at small sample sizes, which implies that these estimators are unstable under multicollinearity for small samples.

5. Conclusion

In this paper, when causal knowledge is available in the form of a Gaussian linear SEM with the corresponding DAG and when the ordinary least squares method is utilized to estimate the total effect, we considered the situation where the causal effect can be estimated based on the back-door criterion. In this situation, we formulated the unbiased estimator of the causal effect on the variance with the exact variance. The estimated causal effect on the variance proposed by Kuroki and Miyakawa (2003) is consistent but not unbiased. For small sample sizes, the use of the consistent estimator in statistical causal inference may lead to misleading findings. To avoid this problem, we showed in Theorem 1 and numerical experiments that the variance estimator, equation (17), performs better than equation (22) for small samples. Theorem 1 would help statistical practitioners to make appropriate predictions about the variation of the outcome variable when conducting external intervention.

Theorem 2 shows that the asymptotic estimation accuracy of the estimated causal effect on the variance depends on the selection of random variables satisfying the back-door criterion; there are some situations where such a difference can be read off from the graph structure without sampling statistical data.

Future work should involve extending our results to (i) a joint intervention that combines several individual interventions and (ii) adaptive control in which the treatment variable is assigned a value based on some variables that are not affected by the treatment variable. Also, the numerical experiments reveal a drawback of the proposed unbiased estimator, namely that it can take a negative value at small sample sizes, when the statistical causal model is not consistent with available data. One suggestion for solving this problem is to use $\max\{0, \hat{\sigma}_{yy|x}\}$ instead of $\hat{\sigma}_{yy|x}$.

Table 3. Causal Path Coefficients

| | Case 1 | | | | Case 2 | | | | Case 3 | | | |
|----------------|--------|--------|----------------|----------------|--------|--------|----------------|----------------|--------|---------|----------------|----------------|
| | Y | X | Z ₁ | Z ₂ | Y | X | Z ₁ | Z ₂ | Y | X | Z ₁ | Z ₂ |
| Y | - | 0.7000 | 0.7000 | 0.0000 | - | 0.7000 | 0.0000 | 0.0000 | - | -0.3430 | 0.7000 | 0.0000 |
| X | - | - | 0.0000 | 0.0000 | - | - | 0.0000 | 0.7000 | - | - | 0.0000 | 0.7000 |
| Z ₁ | - | - | - | 0.7000 | - | - | - | 0.7000 | - | - | - | 0.7000 |

| | Case 4 | | | | Case 5 | | | |
|----------------|--------|--------|----------------|----------------|--------|---------|----------------|----------------|
| | Y | X | Z ₁ | Z ₂ | Y | X | Z ₁ | Z ₂ |
| Y | - | 0.0000 | 0.7000 | 0.0000 | - | -1.9697 | 2.5303 | 0.0000 |
| X | - | - | 0.0000 | 0.7000 | - | - | 0.0000 | 0.9900 |
| Z ₁ | - | - | - | 0.7000 | - | - | - | 0.9900 |

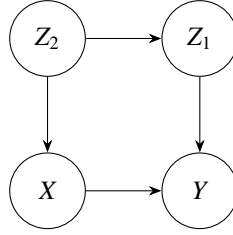


Figure 3. Sample causal path diagram

to evaluate the causal effect on the variance. However, $\max\{0, \hat{\sigma}_{yy|x}\}$ is not an unbiased estimator and it is difficult to formulate the truncated distribution of $\hat{\sigma}_{yy|x}$. Thus, future work would also include developing a more efficient estimator of the causal effect on the variance at small sample sizes. Furthermore, the assumption of Gaussian random variables may be strong. To derive the exact variance formula of the estimated causal effect on the variance for non-Gaussian random variables, our idea for future research is to assume a probability distribution whose exact moments of reciprocal random variables can be derived as explicit expressions. Finally, it would also be necessary to discuss the extension of our result to non-parametric SEMs.

6. Competing interests

The authors declare no conflicts of interest associated with this manuscript.

Acknowledgement

We would like to acknowledge the helpful comments of the associate editor and the two anonymous reviewers. This research was partially supported by Mitsubishi Electric Corporation and Grant-in-Aid for Scientific Research (C) from the Japan Society for the Promotion of Science, Grant Number 19K11856.

Appendix: Proof of Theorem 1

Letting D_x and D_z denote the datasets of X and Z respectively, from the law of total variance (Weiss et al., 2006, pp. 385–386), given $D_x \cup D_z$ we have

$$\text{var}(\hat{\sigma}_{yy|x}) = \text{var}(E(\hat{\sigma}_{yy|x}|D_x, D_z)) + E(\text{var}(\hat{\sigma}_{yy|x}|D_x, D_z)), \quad (33)$$

where $E(\cdot|D_x, D_z)$ and $\text{var}(\cdot|D_x, D_z)$ indicate conditional expectation and variance given $D_x \cup D_z$, respectively. In order to derive the explicit expression of the exact variance formula of the estimated causal effect of $\text{do}(X = x)$ on the

variance $\hat{\sigma}_{yy|x}$ of Y , we calculate the first term $\text{var}(E(\hat{\sigma}_{yy|x}|D_x, D_z))$ and the second term $E(\text{var}(\hat{\sigma}_{yy|x}|D_x, D_z))$ of equation (33) separately.

Step 1: Derivation of $\text{var}(E(\hat{\sigma}_{yy|x}|D_x, D_z))$

Regarding the second term of the right hand side of equation (33), note that we derive

$$\begin{aligned} E(\hat{B}_{yz,xz} \hat{\Sigma}_{zz} \hat{B}'_{yz,xz} | D_x, D_z) &= E(\text{tr}(\hat{\Sigma}_{zz} \hat{B}'_{yz,xz} \hat{B}_{yz,xz}) | D_x, D_z) = \text{tr}(\hat{\Sigma}_{zz} (\sigma_{yy,xz} S_{zz,x}^{-1} + B'_{yz,xz} B_{yz,xz})) \\ &= \sigma_{yy,xz} \text{tr}(\hat{\Sigma}_{zz} S_{zz,x}^{-1}) + B_{yz,xz} \hat{\Sigma}_{zz} B'_{yz,xz} \end{aligned} \quad (34)$$

by Mathai and Provost (1992, p. 53) and the basic formula of the variance-covariance matrix,

$$\begin{aligned} \text{var}(\hat{B}_{yz,xz} | D_x, D_z) &= E(\hat{B}'_{yz,xz} \hat{B}_{yz,xz} | D_x, D_z) - E(\hat{B}'_{yz,xz} | D_x, D_z) E(\hat{B}_{yz,xz} | D_x, D_z) = E(\hat{B}'_{yz,xz} \hat{B}_{yz,xz} | D_x, D_z) - B'_{yz,xz} B_{yz,xz} \\ &= \sigma_{yy,xz} S_{zz,x}^{-1}, \end{aligned} \quad (35)$$

where $\text{tr}(A)$ is the trace or sum of elements on the main diagonal of the square matrix A . Thus, noting that by equation (14), $\hat{\sigma}_{yy,xz}$ is the unbiased estimator of $\sigma_{yy,xz}$, we have

$$\begin{aligned} E(\hat{\sigma}_{yy|x} | D_x, D_z) &= E\left(\hat{\sigma}_{yy,xz} \left(1 - \frac{1}{n-1} \left(q + \frac{\hat{B}_{xz,z} S_{zz} \hat{B}'_{xz,z}}{s_{xx,z}}\right)\right) | D_x, D_z\right) + E(\hat{B}_{yz,xz} \hat{\Sigma}_{zz} \hat{B}'_{yz,xz} | D_x, D_z) \\ &= \sigma_{yy,xz} \left(1 - \frac{1}{n-1} \left(q + \frac{\hat{B}_{xz,z} S_{zz} \hat{B}'_{xz,z}}{s_{xx,z}}\right) + \text{tr}(\hat{\Sigma}_{zz} S_{zz,x}^{-1})\right) + B_{yz,xz} \hat{\Sigma}_{zz} B'_{yz,xz}. \end{aligned} \quad (36)$$

From the Sherman–Morrison formula (Sherman and Morrison, 1950), $S_{zz,x}^{-1}$ can be expressed as

$$S_{zz,x}^{-1} = \left(S_{zz} - \frac{S_{zx} S_{xz}}{s_{xx}}\right)^{-1} = S_{zz}^{-1} + \frac{S_{zz}^{-1} S_{zx} S_{xz} S_{zz}^{-1}}{s_{xx,z}}. \quad (37)$$

Thus, from equation (14) and noting that $\hat{\Sigma}_{zz}$ is the unbiased estimator of Σ_{zz} , we derive

$$\begin{aligned} \text{tr}(\hat{\Sigma}_{zz} S_{zz,x}^{-1}) &= \frac{1}{n-1} \text{tr}(S_{zz} S_{zz,x}^{-1}) = \frac{1}{n-1} \text{tr}\left(I_{q,q} + \frac{S_{zx} S_{xz} S_{zz}^{-1}}{s_{xx,z}}\right) = \frac{1}{n-1} \left(q + \frac{S_{xz} S_{zz}^{-1} S_{zx}}{s_{xx,z}}\right) \\ &= \frac{1}{n-1} \left(q + \frac{\hat{B}_{xz,z} S_{zz} \hat{B}'_{xz,z}}{s_{xx,z}}\right), \end{aligned} \quad (38)$$

where $I_{q,q}$ is the $q \times q$ identity matrix. Thus, since we have

$$E(\hat{\sigma}_{yy|x} | D_x, D_z) = \sigma_{yy,xz} + B_{yz,xz} \hat{\Sigma}_{zz} B'_{yz,xz} \quad (39)$$

from equation (34) as well as equation (38), we derive

$$E(\hat{\sigma}_{yy|x}) = E(E(\hat{\sigma}_{yy|x} | D_x, D_z)) = \sigma_{yy,xz} + B_{yz,xz} \Sigma_{zz} B'_{yz,xz} \quad (40)$$

and

$$\text{var}(E(\hat{\sigma}_{yy|x} | D_x, D_z)) = \text{var}(B_{yz,xz} \hat{\Sigma}_{zz} B'_{yz,xz}). \quad (41)$$

Equation (40) shows that $\hat{\sigma}_{yy|x}$ is the unbiased estimator of the causal effect of $\text{do}(X = x)$ on the variance of Y .

Also, noting that $(n-1)\hat{\Sigma}_{zz}$ follows a Wishart distribution with $n-1$ degrees of freedom and parameter Σ_{zz} and that

$$\frac{(n-1)B_{yz,xz} \hat{\Sigma}_{zz} B'_{yz,xz}}{B_{yz,xz} \Sigma_{zz} B'_{yz,xz}} \quad (42)$$

follows a chi-squared distribution with $n-1$ degrees of freedom (Seber, 2008, p. 466), the variance is given by

$$\text{var}\left(\frac{(n-1)B_{yz,xz} \hat{\Sigma}_{zz} B'_{yz,xz}}{B_{yz,xz} \Sigma_{zz} B'_{yz,xz}}\right) = 2(n-1); \quad (43)$$

that is, we have

$$\text{var}(E(\hat{\sigma}_{yy|x} | D_x, D_z)) = \text{var}\left(B_{yz,xz} \hat{\Sigma}_{zz} B'_{yz,xz}\right) = \frac{2(B_{yz,xz} \Sigma_{zz} B'_{yz,xz})^2}{n-1}. \quad (44)$$

Step 2: Derivation of $E(\text{var}(\hat{\sigma}_{yy|x}|D_x, D_z))$

Noting that $\hat{\sigma}_{yy.xz}$ and $(\hat{\beta}_{yx.xz}, \hat{B}'_{yz.xz})'$ are independent of each other given D_x and D_z (e.g., Mardia et al., 1979), since

$$\frac{(n-q-2)\hat{\sigma}_{yy.xz}}{\sigma_{yy.xz}} \quad (45)$$

follows a chi-squared distribution with $n-q-2$ degrees of freedom, we have

$$\begin{aligned} \text{var}(\hat{\sigma}_{yy|x}|D_x, D_z) &= \text{var}(\hat{\sigma}_{yy.xz}|D_x, D_z) \left(1 - \frac{1}{n-1} \left(q + \frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{s_{xx.z}} \right) \right)^2 + \text{var}(\hat{B}_{yz.xz} \hat{\Sigma}_{zz} \hat{B}'_{yz.xz}|D_x, D_z) \\ &= \frac{2\sigma_{yy.xz}^2}{n-q-2} \left(1 - \frac{1}{n-1} \left(q + \frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{s_{xx.z}} \right) \right)^2 + \text{var}(\hat{B}_{yz.xz} \hat{\Sigma}_{zz} \hat{B}'_{yz.xz}|D_x, D_z) \\ &= \frac{2\sigma_{yy.xz}^2}{n-q-2} \left(\left(1 - \frac{q}{n-1} \right)^2 - 2 \left(1 - \frac{q}{n-1} \right) \frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{(n-1)s_{xx.z}} + \left(\frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{(n-1)s_{xx.z}} \right)^2 \right) + \text{var}(\hat{B}_{yz.xz} \hat{\Sigma}_{zz} \hat{B}'_{yz.xz}|D_x, D_z). \quad (46) \end{aligned}$$

Step 2-1: Derivation of $E\left(\frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{s_{xx.z}}\right)$

Regarding the first term of equation (46), since $\hat{B}_{xz.z}$ and $s_{xx.z}$ are independent of each other given D_z (e.g., Mardia et al, 1979) and noting that $s_{xx.z}/\sigma_{xx.z}$ follows a chi-squared distribution with $n-q-1$ degrees of freedom, we have

$$E\left(\frac{1}{s_{xx.z}}|D_z\right) = \frac{1}{(n-q-3)\sigma_{xx.z}}. \quad (47)$$

Thus, we have

$$\begin{aligned} E\left(\frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{s_{xx.z}}\right) &= E\left(E\left(\frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{s_{xx.z}}|D_z\right)\right) = E\left(E(\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}|D_z) E\left(\frac{1}{s_{xx.z}}|D_z\right)\right) \\ &= \frac{\sigma_{xx.z} E(\text{tr}(S_{zz} S_{zz}^{-1})) + B_{xz.z} E(S_{zz}) B'_{xz.z}}{(n-q-3)\sigma_{xx.z}} = \frac{q\sigma_{xx.z} + (n-1)B_{xz.z} \Sigma_{zz} B'_{xz.z}}{(n-q-3)\sigma_{xx.z}} \quad (48) \end{aligned}$$

from

$$\text{var}(\hat{B}_{xz.z}) = E(\hat{B}'_{xz.z} \hat{B}_{xz.z}) - B'_{xz.z} B_{xz.z} = \sigma_{xx.z} S_{zz}^{-1}. \quad (49)$$

Step 2-2: Derivation of $E\left(\left(\frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{s_{xx.z}}\right)^2\right)$

Similarly to Step 2-1, from

$$E\left(\frac{1}{s_{xx.z}^2}\right) = \frac{1}{(n-q-3)(n-q-5)\sigma_{xx.z}^2}, \quad (50)$$

since $\hat{B}_{xz.z}$ and $\hat{\sigma}_{xx.z}$ are independent of each other given D_z (e.g., Mardia et al., 1979) we derive

$$\begin{aligned} E\left(\left(\frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{s_{xx.z}}\right)^2\right) &= E\left(E\left(\left(\frac{\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}}{s_{xx.z}}\right)^2|D_z\right)\right) = E\left(E((\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z})^2|D_z) E\left(\frac{1}{s_{xx.z}^2}|D_z\right)\right) \\ &= \frac{E(E((\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z})^2|D_z))}{(n-q-3)(n-q-5)\sigma_{xx.z}^2} = \frac{E(\text{var}(\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}|D_z))}{(n-q-3)(n-q-5)\sigma_{xx.z}^2} + \frac{E(E(\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}|D_z)^2)}{(n-q-3)(n-q-5)\sigma_{xx.z}^2}. \quad (51) \end{aligned}$$

From Seber (2008, p. 438), $E(\text{var}(\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}|D_z))$ is given by

$$\begin{aligned} E(\text{var}(\hat{B}_{xz.z} S_{zz} \hat{B}'_{xz.z}|D_z)) &= 2\sigma_{xx.z}^2 E(\text{tr}(S_{zz} S_{zz}^{-1} S_{zz} S_{zz}^{-1})) + 4E(\sigma_{xx.z} B_{xz.z} S_{zz} S_{zz}^{-1} S_{zz} B'_{xz.z}) \\ &= 2q\sigma_{xx.z}^2 + 4(n-1)\sigma_{xx.z} B_{xz.z} \Sigma_{zz} B'_{xz.z}. \quad (52) \end{aligned}$$

Again, from

$$\text{var}(B_{xz,z}S_{zz}B'_{xz,z}) = 2(n-1)(B_{xz,z}\Sigma_{zz}B'_{xz,z})^2 \quad (53)$$

(Seber, 2008, p. 466), we have

$$\begin{aligned} E\left(\left(\frac{\hat{B}_{xz,z}S_{zz}\hat{B}'_{xz,z}}{s_{xx,z}}\right)^2\right) &= E\left(E\left(\left(\frac{\hat{B}_{xz,z}S_{zz}\hat{B}'_{xz,z}}{s_{xx,z}}\right)^2 \middle| D_z\right)\right) \\ &= \frac{2q\sigma_{xx,z}^2 + 4(n-1)\sigma_{xx,z}B_{xz,z}\Sigma_{zz}B'_{xz,z}}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} + \frac{E((q\sigma_{xx,z} + B_{xz,z}S_{zz}B'_{xz,z})^2)}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} \\ &= \frac{2q\sigma_{xx,z}^2 + 4(n-1)\sigma_{xx,z}B_{xz,z}\Sigma_{zz}B'_{xz,z}}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} + \frac{\text{var}(B_{xz,z}S_{zz}B'_{xz,z})}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} + \frac{E(q\sigma_{xx,z} + B_{xz,z}S_{zz}B'_{xz,z})^2}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} \\ &= \frac{2q\sigma_{xx,z}^2 + 4(n-1)\sigma_{xx,z}B_{xz,z}\Sigma_{zz}B'_{xz,z}}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} + \frac{2(n-1)(B_{xz,z}\Sigma_{zz}B'_{xz,z})^2}{(n-q-3)(n-q-5)\sigma_{xx,z}^2} + \frac{(q\sigma_{xx,z} + (n-1)B_{xz,z}\Sigma_{zz}B'_{xz,z})^2}{(n-q-3)(n-q-5)\sigma_{xx,z}^2}. \end{aligned} \quad (54)$$

Step 2-3: Derivation of $\text{var}(\hat{B}_{yz,xz}\hat{\Sigma}_{zz}\hat{B}'_{yz,xz}|D_x, D_z)$

Regarding the second term of equation (46), from Mathai and Provost (1992, p. 53) we have

$$\text{var}(\hat{B}_{yz,xz}\hat{\Sigma}_{zz}\hat{B}'_{yz,xz}|D_x, D_z) = \frac{2\sigma_{yy,xz}^2}{(n-1)^2}\text{tr}(S_{zz}S_{zz,x}^{-1}S_{zz}S_{zz,x}^{-1}) + \frac{4\sigma_{yy,xz}}{(n-1)^2}B_{yz,xz}S_{zz}S_{zz,x}^{-1}S_{zz}B'_{yz,xz}. \quad (55)$$

From equation (37) and $\hat{B}_{xz,z} = S_{xz}S_{zz}^{-1}$,

$$\begin{aligned} E(\text{var}(\hat{B}_{yz,xz}\hat{\Sigma}_{zz}\hat{B}'_{yz,xz}|D_x, D_z)) &= E\left(\frac{2\sigma_{yy,xz}^2}{(n-1)^2}\text{tr}\left(\left(S_{zz} + \frac{S_{zx}S_{xz}}{s_{xx,z}}\right)S_{zz,x}^{-1}\right)\right) + \frac{4\sigma_{yy,xz}}{(n-1)^2}B_{yz,xz}E\left(S_{zz} + \frac{S_{zx}S_{xz}}{s_{xx,z}}\right)B'_{yz,xz} \\ &= \frac{2\sigma_{yy,xz}^2}{(n-1)^2}\left(q + 2E\left(\frac{\hat{B}_{xz,z}S_{zz}\hat{B}'_{xz,z}}{s_{xx,z}}\right) + E\left(\left(\frac{\hat{B}_{xz,z}S_{zz}\hat{B}'_{xz,z}}{s_{xx,z}}\right)^2\right)\right) \\ &\quad + \frac{4\sigma_{yy,xz}}{(n-1)^2}\left((n-1)B_{yz,xz}\Sigma_{zz}B'_{yz,xz} + E\left(\frac{(B_{yz,xz}S_{zz}B'_{xz,z})^2}{s_{xx,z}}\right)\right). \end{aligned} \quad (56)$$

Now, from the law of total variance (Weiss et al, 2006, pp. 385–386), we have

$$\begin{aligned} E\left(E\left((B_{yz,xz}S_{zz}\hat{B}'_{xz,z})^2 \middle| D_z\right)\right) &= E\left(\text{var}\left(B_{yz,xz}S_{zz}\hat{B}'_{xz,z} \middle| D_z\right) + E\left((B_{yz,xz}S_{zz}\hat{B}'_{xz,z})^2 \middle| D_z\right)\right) \\ &= \sigma_{xx,z}B_{yz,xz}E(S_{zz})B'_{yz,xz} + E((B_{yz,xz}S_{zz}B'_{xz,z})^2) = (n-1)\sigma_{xx,z}B_{yz,xz}\Sigma_{zz}B'_{yz,xz} + E((B_{yz,xz}S_{zz}B'_{xz,z})^2). \end{aligned} \quad (57)$$

Finally, from Seber (2008, p. 467), we have

$$\begin{aligned} E((B_{yz,xz}S_{zz}B'_{xz,z})^2) &= B_{yz,xz}E(S_{zz}B'_{xz,z}B_{xz,z}S_{zz})B'_{yz,xz} \\ &= ((n-1) + (n-1)^2)(B_{yz,xz}\Sigma_{zz}B'_{xz,z})^2 + (n-1)(B_{xz,z}\Sigma_{zz}B'_{xz,z})(B_{yz,xz}B'_{xz,z})^2. \end{aligned} \quad (58)$$

Based on the above derivation, we derive the exact variance formula of the estimated causal effect of $\text{do}(X = x)$ on the variance of Y .

References

- [1] Baron, R. and Kenny, D. (1986). The moderator-mediator variable distinction in social psychological research: Conceptual, strategic, and statistical considerations. *Journal of Personality and Social Psychology*, **51**, 1173–1182.
- [2] Bollen, K. A. (1987). Total, direct, and indirect effects in structural equation models. *Sociological Methodology*, **17**, 37–69.
- [3] Bollen, K. A. (1989). *Structural Equations with Latent Variables*, John Wiley & Sons.
- [4] Brito, C. (2004). Graphical methods for identification in structural equation models. Ph.D. Thesis, Department of Computer Science Department, UCLA.

- [5] Chan, H. and Kuroki, M. (2010). Using descendants as instrumental variables for the identification of direct causal effects in linear SEMs. *Proceedings of the 13th International Conference on Artificial Intelligence and Statistics*, 73-80.
- [6] Chen, B. R. (2017). Graphical methods for linear structural equation modeling. Ph.D. Thesis, Department of Computer Science Department, UCLA.
- [7] Cox, D. R. and Wermuth, N. (2014). *Multivariate Dependencies: Models, Analysis and Interpretation*, Chapman and Hall/CRC.
- [8] Duncan, O. (1975). *Introduction to Structural Equation Models*. Academic Press, New York.
- [9] Gische, C., West, S. G. and Voelkle, M. C. (2021). Forecasting causal effects of interventions versus predicting future outcomes. *Structural Equation Modeling: A Multidisciplinary Journal*, **28**, 475-492.
- [10] Henckel, L., Perković, E. and Maathuis, M. H. (2019). Graphical criteria for efficient total effect estimation via adjustment in causal linear models. arXiv preprint arXiv:1907.02435.
- [11] Hernán, M. A. and Robins, J. M. (2020). *Causal Inference: What If*, Chapman & Hall/CRC.
- [12] Kuroki, M. (2008). The evaluation of causal effects on the variance and its application to process analysis. *Journal of the Japanese Society for Quality Control*, **38**, 373-384.
- [13] Kuroki, M. (2012). Optimizing an external intervention using a structural equation model with an application to statistical process analysis. *Journal of Applied Statistics*, **39**, 673-694.
- [14] Kuroki, M. and Cai, Z. (2004). Selection of identifiability criteria for total effects by using path diagrams. *Proceedings of the 20th Conference on Uncertainty in Artificial Intelligence*, 333-340.
- [15] Kuroki, M. and Miyakawa, M. (1999a). Estimation of causal effects in causal diagrams and its application to process analysis. *Journal of the Japanese Society for Quality Control*, **29**, 70-80.
- [16] Kuroki, M. and Miyakawa, M. (1999b). Estimation of conditional intervention effect in adaptive control. *Journal of the Japanese Society for Quality Control*, **29**, 237-245.
- [17] Kuroki, M. and Miyakawa, M. (2003). Covariate selection for estimating the causal effect of external interventions using causal diagrams. *Journal of the Royal Statistical Society, Series B*, **65**, 209-222.
- [18] Kuroki, M. and Nanmo, H. (2020). Variance formulas for estimated causal effect on the mean and predicted outcome with external intervention based on the back-door criterion in linear structural equation models. *AStA Advances in Statistical Analysis*, **104**, 667-685.
- [19] Kuroki, M. and Pearl, J. (2014). Measurement bias and effect restoration in causal inference. *Biometrika*, **101**, 423-437.
- [20] Maathuis, M. H. and Colombo, D. (2015). A generalized back-door criterion. *Annals of Statistics*, **43**, 1060-1088.
- [21] Mardia, K. V., Kent, J. T., and Bibby, J. M. (1979). *Multivariate Analysis*, Academic Press.
- [22] Mathai, A. M. and Provost, S. B. (1992). *Quadratic Forms in Random Variables: Theory and Applications*, CRC Press.
- [23] Nandy, P., Maathuis, M. H. and Richardson, T. S. (2017). Estimating the effect of joint interventions from observational data in sparse high-dimensional settings. *Annals of Statistics*, **45**, 647-674.
- [24] Okuno, T., Katayama, Z., Kamigori, N., Itoh, T., Irikura, N., and Fujiwara, N. (1986). *Multivariate Data Analysis in Industry*, JUSE Press.
- [25] Pearl, J. (2009). *Causality: Models, Reasoning, and Inference, the 2nd Edition*, Cambridge University Press.
- [26] Perković, E. (2018). Graphical characterizations of adjustment sets. Ph.D. Thesis, Department of Mathematics, ETH Zurich.
- [27] Seber, G. A. (2008). *A Matrix Handbook for Statisticians*, John Wiley & Sons.
- [28] Shan, N. and Guo, J. (2010). Covariate selection for identifying the effects of a particular type of conditional plan using causal networks. *Frontiers of Mathematics in China*, **5**, 687-700.
- [29] Shan, N. and Guo, J. (2012). Covariate selection for identifying the causal effects of stochastic interventions using causal networks. *Journal of Statistical Planning and Inference*, **142**, 212-220.
- [30] Sherman, J. and Morrison, W. J. (1950). Adjustment of an inverse matrix corresponding to a change in one element of a given matrix. *Annals of Mathematical Statistics*, **21**, 124-127.
- [31] Tian, J. (2004). Identifying linear causal effects. *Proceeding of the 19th National Conference on Artificial Intelligence*, 104-111.
- [32] Weiss, N. A., Holmes, P. T. and Hardy, M. (2006). *A Course in Probability*. Pearson Addison Wesley.
- [33] Wright, S. (1923). The theory of path coefficients: A reply to Niles' criticism. *Genetics*, **8**, 239-255.
- [34] Wright, S. (1934). The method of path coefficients. *Annals of Mathematical Statistics*, **5**, 161-215.

Table 4. Numerical Experiments -Normal Distribution-.

Case 1: $\sigma_{y|jk} = 0.510$

| Sample size | 10 | | | | | | 25 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.509 | 0.511 | 0.509 | 0.544 | 0.509 | 0.514 | 0.510 | 0.511 | 0.510 | 0.522 | 0.510 | 0.511 |
| Equation (17)/(22) | 0.059 | 0.052 | 0.065 | 0.052 | 0.059 | 0.052 | 0.022 | 0.021 | 0.022 | 0.021 | 0.022 | 0.021 |
| var | 0.058 | 0.058 | 0.064 | 0.069 | 0.058 | 0.058 | 0.022 | 0.022 | 0.023 | 0.023 | 0.022 | 0.022 |
| Skewness | 0.937 | 0.936 | 1.038 | 1.000 | 0.937 | 0.936 | 0.570 | 0.570 | 0.576 | 0.566 | 0.571 | 0.570 |
| Kurtosis | 4.326 | 4.325 | 4.908 | 4.780 | 4.328 | 4.327 | 3.490 | 3.490 | 3.481 | 3.463 | 3.492 | 3.492 |
| Minimum | 0.016 | 0.017 | 0.016 | 0.018 | 0.016 | 0.019 | 0.102 | 0.103 | 0.105 | 0.108 | 0.100 | 0.102 |
| 1st Quartile | 0.333 | 0.335 | 0.324 | 0.352 | 0.333 | 0.338 | 0.404 | 0.405 | 0.402 | 0.413 | 0.404 | 0.406 |
| Median | 0.472 | 0.474 | 0.467 | 0.501 | 0.472 | 0.477 | 0.496 | 0.497 | 0.495 | 0.507 | 0.495 | 0.497 |
| 3rd Quartile | 0.644 | 0.647 | 0.649 | 0.691 | 0.644 | 0.650 | 0.601 | 0.601 | 0.603 | 0.615 | 0.601 | 0.603 |
| Maximum | 2.053 | 2.056 | 2.995 | 3.071 | 2.052 | 2.057 | 1.344 | 1.345 | 1.371 | 1.387 | 1.344 | 1.346 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.511 | 0.511 | 0.511 | 0.516 | 0.511 | 0.511 | 0.509 | 0.510 | 0.509 | 0.512 | 0.509 | 0.510 |
| Equation (17)/(22) | 0.011 | 0.010 | 0.011 | 0.010 | 0.011 | 0.010 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 |
| var | 0.011 | 0.011 | 0.011 | 0.011 | 0.011 | 0.011 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 |
| Skewness | 0.407 | 0.407 | 0.411 | 0.407 | 0.408 | 0.407 | 0.300 | 0.300 | 0.301 | 0.300 | 0.300 | 0.300 |
| Kurtosis | 3.237 | 3.237 | 3.238 | 3.233 | 3.239 | 3.239 | 3.110 | 3.110 | 3.115 | 3.114 | 3.110 | 3.110 |
| Minimum | 0.210 | 0.211 | 0.205 | 0.209 | 0.210 | 0.211 | 0.265 | 0.265 | 0.266 | 0.268 | 0.265 | 0.265 |
| 1st Quartile | 0.438 | 0.438 | 0.437 | 0.442 | 0.438 | 0.438 | 0.459 | 0.459 | 0.459 | 0.461 | 0.459 | 0.459 |
| Median | 0.504 | 0.504 | 0.504 | 0.509 | 0.504 | 0.504 | 0.506 | 0.506 | 0.506 | 0.508 | 0.506 | 0.506 |
| 3rd Quartile | 0.576 | 0.576 | 0.576 | 0.582 | 0.576 | 0.577 | 0.557 | 0.557 | 0.557 | 0.559 | 0.557 | 0.557 |
| Maximum | 1.034 | 1.035 | 1.044 | 1.053 | 1.035 | 1.035 | 0.867 | 0.867 | 0.876 | 0.880 | 0.867 | 0.867 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 |
| Equation (17)/(22) | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| var | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Skewness | 0.075 | 0.075 | 0.075 | 0.075 | 0.075 | 0.075 | 0.052 | 0.052 | 0.052 | 0.052 | 0.052 | 0.052 |
| Kurtosis | 3.041 | 3.041 | 3.037 | 3.037 | 3.041 | 3.041 | 3.030 | 3.030 | 3.030 | 3.030 | 3.030 | 3.030 |
| Minimum | 0.392 | 0.392 | 0.393 | 0.393 | 0.392 | 0.392 | 0.432 | 0.432 | 0.432 | 0.432 | 0.432 | 0.432 |
| 1st Quartile | 0.491 | 0.491 | 0.491 | 0.492 | 0.491 | 0.491 | 0.497 | 0.497 | 0.497 | 0.497 | 0.497 | 0.497 |
| Median | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 |
| 3rd Quartile | 0.528 | 0.528 | 0.528 | 0.529 | 0.528 | 0.528 | 0.523 | 0.523 | 0.523 | 0.523 | 0.523 | 0.523 |
| Maximum | 0.651 | 0.651 | 0.652 | 0.652 | 0.651 | 0.651 | 0.593 | 0.593 | 0.594 | 0.594 | 0.593 | 0.593 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 4. Numerical Experiments -Normal Distribution-.

Case 2: $\sigma_{y|jk} = 0.510$

| Sample size | 10 | | | | | | 25 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.603 | 0.510 | 0.658 | 0.511 | 0.744 | 0.511 | 0.541 | 0.511 | 0.556 | 0.511 | 0.580 |
| Equation (17)/(22) | 0.073 | 0.052 | 0.107 | 0.052 | 0.145 | 0.052 | 0.023 | 0.021 | 0.024 | 0.021 | 0.025 | 0.021 |
| var | 0.073 | 0.099 | 0.110 | 0.155 | 0.145 | 0.221 | 0.023 | 0.026 | 0.024 | 0.028 | 0.025 | 0.031 |
| Skewness | 1.478 | 1.531 | 4.255 | 4.000 | 4.826 | 4.528 | 0.620 | 0.617 | 0.709 | 0.697 | 0.748 | 0.714 |
| Kurtosis | 10.112 | 11.673 | 87.363 | 74.588 | 94.755 | 74.449 | 3.688 | 3.673 | 3.964 | 3.932 | 4.234 | 4.070 |
| Minimum | -0.814 | 0.022 | -2.621 | 0.022 | -5.144 | 0.026 | 0.108 | 0.113 | 0.107 | 0.115 | 0.107 | 0.120 |
| 1st Quartile | 0.318 | 0.377 | 0.300 | 0.400 | 0.288 | 0.446 | 0.403 | 0.427 | 0.400 | 0.436 | 0.399 | 0.455 |
| Median | 0.464 | 0.548 | 0.450 | 0.585 | 0.443 | 0.655 | 0.496 | 0.525 | 0.494 | 0.538 | 0.494 | 0.561 |
| 3rd Quartile | 0.652 | 0.769 | 0.646 | 0.827 | 0.648 | 0.935 | 0.602 | 0.637 | 0.602 | 0.655 | 0.602 | 0.683 |
| Maximum | 5.069 | 6.851 | 13.617 | 15.156 | 13.302 | 15.291 | 1.624 | 1.706 | 1.653 | 1.756 | 1.940 | 2.073 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.524 | 0.510 | 0.531 | 0.510 | 0.542 | 0.510 | 0.517 | 0.510 | 0.520 | 0.510 | 0.525 |
| Equation (17)/(22) | 0.011 | 0.010 | 0.011 | 0.010 | 0.011 | 0.010 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 |
| var | 0.011 | 0.011 | 0.011 | 0.012 | 0.011 | 0.012 | 0.005 | 0.005 | 0.005 | 0.006 | 0.005 | 0.006 |
| Skewness | 0.390 | 0.391 | 0.416 | 0.414 | 0.423 | 0.418 | 0.300 | 0.300 | 0.310 | 0.309 | 0.310 | 0.308 |
| Kurtosis | 3.206 | 3.207 | 3.270 | 3.267 | 3.297 | 3.288 | 3.152 | 3.152 | 3.187 | 3.185 | 3.185 | 3.183 |
| Minimum | 0.193 | 0.198 | 0.186 | 0.196 | 0.186 | 0.199 | 0.247 | 0.250 | 0.245 | 0.250 | 0.245 | 0.253 |
| 1st Quartile | 0.436 | 0.448 | 0.435 | 0.454 | 0.435 | 0.463 | 0.459 | 0.465 | 0.458 | 0.468 | 0.458 | 0.472 |
| Median | 0.503 | 0.516 | 0.502 | 0.523 | 0.502 | 0.534 | 0.506 | 0.513 | 0.506 | 0.516 | 0.506 | 0.522 |
| 3rd Quartile | 0.576 | 0.592 | 0.577 | 0.601 | 0.577 | 0.613 | 0.557 | 0.565 | 0.557 | 0.568 | 0.557 | 0.574 |
| Maximum | 0.994 | 1.026 | 1.071 | 1.113 | 1.110 | 1.171 | 0.898 | 0.910 | 1.010 | 1.030 | 1.009 | 1.037 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.511 | 0.510 | 0.512 | 0.510 | 0.513 | 0.510 | 0.511 | 0.510 | 0.511 | 0.510 | 0.512 |
| Equation (17)/(22) | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| var | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Skewness | 0.028 | 0.028 | 0.029 | 0.029 | 0.029 | 0.029 | 0.024 | 0.024 | 0.025 | 0.025 | 0.025 | 0.025 |
| Kurtosis | 2.981 | 2.981 | 2.981 | 2.980 | 2.980 | 2.980 | 3.001 | 3.001 | 2.999 | 2.999 | 3.000 | 2.999 |
| Minimum | 0.427 | 0.428 | 0.427 | 0.428 | 0.427 | 0.429 | 0.455 | 0.455 | 0.455 | 0.455 | 0.455 | 0.456 |
| 1st Quartile | 0.496 | 0.497 | 0.496 | 0.498 | 0.496 | 0.499 | 0.500 | 0.501 | 0.500 | 0.501 | 0.500 | 0.502 |
| Median | 0.510 | 0.511 | 0.510 | 0.512 | 0.510 | 0.513 | 0.510 | 0.511 | 0.510 | 0.511 | 0.510 | 0.511 |
| 3rd Quartile | 0.524 | 0.525 | 0.524 | 0.526 | 0.524 | 0.527 | 0.520 | 0.521 | 0.520 | 0.521 | 0.520 | 0.521 |
| Maximum | 0.601 | 0.603 | 0.600 | 0.603 | 0.600 | 0.604 | 0.577 | 0.578 | 0.577 | 0.578 | 0.577 | 0.578 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 4. Numerical Experiments -Normal Distribution-.

Case 3: $\sigma_{y|jk} = 1.118$

| Sample size | 10 | | | | | | 25 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.119 | 1.234 | 1.119 | 1.373 | 1.120 | 1.409 | 1.116 | 1.153 | 1.118 | 1.196 | 1.117 | 1.202 |
| Equation (17)/(22) | 0.397 | 0.289 | 0.596 | 0.331 | 0.566 | 0.308 | 0.128 | 0.115 | 0.154 | 0.132 | 0.141 | 0.123 |
| var | 0.408 | 0.457 | 0.584 | 0.725 | 0.595 | 0.721 | 0.130 | 0.135 | 0.157 | 0.171 | 0.145 | 0.156 |
| Skewness | 1.837 | 1.873 | 2.391 | 2.328 | 2.678 | 2.774 | 0.751 | 0.745 | 1.040 | 1.006 | 0.895 | 0.867 |
| Kurtosis | 14.121 | 15.044 | 15.610 | 14.888 | 24.310 | 24.522 | 4.008 | 4.012 | 5.210 | 5.073 | 4.491 | 4.418 |
| Minimum | -0.420 | 0.043 | -2.802 | 0.043 | -8.253 | 0.051 | 0.161 | 0.179 | 0.168 | 0.184 | 0.161 | 0.191 |
| 1st Quartile | 0.668 | 0.758 | 0.617 | 0.804 | 0.620 | 0.849 | 0.860 | 0.892 | 0.838 | 0.903 | 0.846 | 0.921 |
| Median | 0.999 | 1.108 | 0.952 | 1.194 | 0.964 | 1.238 | 1.074 | 1.110 | 1.060 | 1.138 | 1.065 | 1.149 |
| 3rd Quartile | 1.433 | 1.568 | 1.421 | 1.721 | 1.427 | 1.756 | 1.328 | 1.370 | 1.333 | 1.422 | 1.331 | 1.425 |
| Maximum | 13.817 | 14.935 | 12.257 | 13.943 | 17.465 | 18.915 | 3.792 | 3.959 | 4.504 | 4.658 | 4.427 | 4.560 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.117 | 1.134 | 1.117 | 1.153 | 1.117 | 1.156 | 1.119 | 1.127 | 1.119 | 1.136 | 1.119 | 1.138 |
| Equation (17)/(22) | 0.061 | 0.058 | 0.071 | 0.066 | 0.065 | 0.062 | 0.030 | 0.029 | 0.034 | 0.033 | 0.032 | 0.031 |
| var | 0.061 | 0.062 | 0.071 | 0.074 | 0.066 | 0.069 | 0.030 | 0.030 | 0.034 | 0.035 | 0.032 | 0.032 |
| Skewness | 0.516 | 0.513 | 0.679 | 0.670 | 0.611 | 0.600 | 0.337 | 0.336 | 0.443 | 0.439 | 0.392 | 0.388 |
| Kurtosis | 3.461 | 3.457 | 3.898 | 3.881 | 3.718 | 3.697 | 3.184 | 3.184 | 3.342 | 3.337 | 3.277 | 3.273 |
| Minimum | 0.337 | 0.344 | 0.340 | 0.360 | 0.332 | 0.357 | 0.561 | 0.567 | 0.555 | 0.568 | 0.575 | 0.586 |
| 1st Quartile | 0.943 | 0.959 | 0.929 | 0.962 | 0.935 | 0.971 | 0.998 | 1.006 | 0.988 | 1.005 | 0.993 | 1.011 |
| Median | 1.096 | 1.114 | 1.088 | 1.125 | 1.092 | 1.132 | 1.110 | 1.118 | 1.105 | 1.123 | 1.108 | 1.127 |
| 3rd Quartile | 1.269 | 1.288 | 1.274 | 1.314 | 1.272 | 1.315 | 1.229 | 1.238 | 1.234 | 1.253 | 1.231 | 1.251 |
| Maximum | 2.435 | 2.470 | 3.173 | 3.241 | 2.545 | 2.609 | 2.075 | 2.091 | 2.111 | 2.133 | 2.071 | 2.092 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.117 | 1.119 | 1.118 | 1.121 | 1.117 | 1.121 | 1.118 | 1.118 | 1.118 | 1.119 | 1.118 | 1.120 |
| Equation (17)/(22) | 0.006 | 0.006 | 0.007 | 0.007 | 0.006 | 0.006 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 |
| var | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.002 | 0.002 | 0.003 | 0.003 | 0.002 | 0.002 |
| Skewness | 0.100 | 0.100 | 0.155 | 0.154 | 0.129 | 0.128 | 0.097 | 0.097 | 0.117 | 0.117 | 0.111 | 0.111 |
| Kurtosis | 2.994 | 2.994 | 3.001 | 3.001 | 2.993 | 2.993 | 3.026 | 3.026 | 3.053 | 3.053 | 3.040 | 3.040 |
| Minimum | 0.860 | 0.861 | 0.844 | 0.847 | 0.848 | 0.852 | 0.932 | 0.933 | 0.909 | 0.911 | 0.922 | 0.923 |
| 1st Quartile | 1.071 | 1.073 | 1.066 | 1.070 | 1.069 | 1.073 | 1.085 | 1.086 | 1.082 | 1.084 | 1.084 | 1.085 |
| Median | 1.116 | 1.118 | 1.116 | 1.119 | 1.116 | 1.120 | 1.117 | 1.118 | 1.116 | 1.118 | 1.117 | 1.119 |
| 3rd Quartile | 1.163 | 1.164 | 1.166 | 1.170 | 1.164 | 1.168 | 1.149 | 1.150 | 1.152 | 1.154 | 1.151 | 1.153 |
| Maximum | 1.416 | 1.418 | 1.482 | 1.486 | 1.434 | 1.438 | 1.316 | 1.316 | 1.348 | 1.350 | 1.342 | 1.344 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 4. Numerical Experiments -Normal Distribution-.

Case 4: $\sigma_{y|jk} = 1.000$

| Sample size | 10 | | | | | | 25 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.001 | 1.094 | 0.999 | 1.220 | 0.999 | 1.233 | 1.000 | 1.030 | 0.999 | 1.067 | 1.000 | 1.069 |
| Equation (17)/(22) | 0.316 | 0.232 | 0.481 | 0.270 | 0.436 | 0.247 | 0.102 | 0.093 | 0.126 | 0.108 | 0.113 | 0.099 |
| var | 0.317 | 0.349 | 0.483 | 0.593 | 0.463 | 0.526 | 0.102 | 0.105 | 0.124 | 0.135 | 0.113 | 0.120 |
| Skewness | 1.444 | 1.449 | 3.284 | 3.150 | 0.595 | 2.589 | 0.731 | 0.726 | 1.016 | 0.988 | 0.861 | 0.840 |
| Kurtosis | 7.138 | 7.186 | 39.242 | 34.466 | 96.540 | 22.520 | 3.903 | 3.899 | 5.103 | 5.015 | 4.351 | 4.314 |
| Minimum | -2.116 | 0.038 | -3.755 | 0.040 | -29.146 | 0.051 | 0.164 | 0.170 | 0.161 | 0.175 | 0.157 | 0.176 |
| 1st Quartile | 0.603 | 0.678 | 0.553 | 0.718 | 0.563 | 0.753 | 0.770 | 0.797 | 0.749 | 0.806 | 0.760 | 0.821 |
| Median | 0.897 | 0.985 | 0.851 | 1.061 | 0.868 | 1.092 | 0.964 | 0.993 | 0.948 | 1.015 | 0.955 | 1.024 |
| 3rd Quartile | 1.278 | 1.384 | 1.270 | 1.529 | 1.276 | 1.539 | 1.189 | 1.222 | 1.193 | 1.270 | 1.191 | 1.268 |
| Maximum | 7.246 | 7.902 | 18.483 | 19.766 | 13.702 | 15.580 | 3.042 | 3.114 | 4.030 | 4.203 | 3.577 | 3.719 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.000 | 1.015 | 1.000 | 1.032 | 1.001 | 1.033 | 1.000 | 1.007 | 1.000 | 1.015 | 1.000 | 1.016 |
| Equation (17)/(22) | 0.049 | 0.046 | 0.058 | 0.054 | 0.052 | 0.049 | 0.024 | 0.023 | 0.028 | 0.027 | 0.025 | 0.025 |
| var | 0.049 | 0.050 | 0.059 | 0.061 | 0.054 | 0.055 | 0.024 | 0.024 | 0.028 | 0.029 | 0.026 | 0.026 |
| Skewness | 0.515 | 0.512 | 0.677 | 0.669 | 0.601 | 0.592 | 0.360 | 0.359 | 0.461 | 0.457 | 0.412 | 0.409 |
| Kurtosis | 3.500 | 3.498 | 3.871 | 3.854 | 3.709 | 3.690 | 3.240 | 3.239 | 3.407 | 3.402 | 3.324 | 3.321 |
| Minimum | 0.324 | 0.330 | 0.313 | 0.329 | 0.312 | 0.330 | 0.505 | 0.510 | 0.481 | 0.490 | 0.488 | 0.499 |
| 1st Quartile | 0.844 | 0.857 | 0.829 | 0.857 | 0.837 | 0.867 | 0.892 | 0.898 | 0.881 | 0.895 | 0.887 | 0.902 |
| Median | 0.981 | 0.995 | 0.975 | 1.006 | 0.979 | 1.011 | 0.991 | 0.998 | 0.988 | 1.003 | 0.989 | 1.005 |
| 3rd Quartile | 1.139 | 1.154 | 1.144 | 1.179 | 1.141 | 1.176 | 1.099 | 1.107 | 1.105 | 1.122 | 1.102 | 1.118 |
| Maximum | 2.508 | 2.532 | 2.559 | 2.590 | 2.419 | 2.447 | 1.803 | 1.814 | 2.020 | 2.044 | 1.877 | 1.895 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.000 | 1.001 | 1.000 | 1.003 | 1.000 | 1.003 | 1.000 | 1.001 | 1.000 | 1.002 | 1.000 | 1.002 |
| Equation (17)/(22) | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.002 | 0.002 | 0.003 | 0.003 | 0.002 | 0.002 |
| var | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 |
| Skewness | 0.125 | 0.125 | 0.173 | 0.173 | 0.142 | 0.141 | 0.089 | 0.089 | 0.123 | 0.123 | 0.107 | 0.107 |
| Kurtosis | 3.052 | 3.052 | 3.063 | 3.063 | 3.050 | 3.050 | 2.982 | 2.982 | 3.023 | 3.023 | 3.005 | 3.005 |
| Minimum | 0.773 | 0.774 | 0.752 | 0.754 | 0.768 | 0.770 | 0.825 | 0.825 | 0.797 | 0.799 | 0.815 | 0.816 |
| 1st Quartile | 0.958 | 0.960 | 0.954 | 0.957 | 0.956 | 0.959 | 0.970 | 0.971 | 0.967 | 0.969 | 0.969 | 0.970 |
| Median | 0.999 | 1.000 | 0.998 | 1.001 | 0.998 | 1.001 | 0.999 | 1.000 | 0.999 | 1.000 | 0.999 | 1.001 |
| 3rd Quartile | 1.040 | 1.041 | 1.043 | 1.046 | 1.042 | 1.045 | 1.029 | 1.030 | 1.031 | 1.033 | 1.030 | 1.031 |
| Maximum | 1.285 | 1.287 | 1.299 | 1.303 | 1.291 | 1.294 | 1.191 | 1.191 | 1.235 | 1.236 | 1.224 | 1.225 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 4. Numerical Experiments -Normal Distribution-.

Case 5: $\sigma_{y|x} = 6.890$

| Sample size | 10 | | | | | | 25 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 6.889 | 8.935 | 6.846 | 11.998 | 6.818 | 11.758 | 6.891 | 7.478 | 6.839 | 8.310 | 6.849 | 8.093 |
| Equation (17)/(22) | 89.284 | 39.921 | 273.079 | 85.493 | 258.651 | 69.743 | 20.461 | 15.968 | 48.386 | 34.197 | 39.017 | 27.897 |
| var | 89.195 | 96.603 | 269.086 | 288.518 | 261.860 | 276.373 | 20.314 | 20.997 | 48.615 | 50.410 | 39.916 | 41.375 |
| Skewness | 3.174 | 3.590 | 3.751 | 4.816 | 3.909 | 5.284 | 1.404 | 1.435 | 1.944 | 2.010 | 1.797 | 1.864 |
| Kurtosis | 27.518 | 32.126 | 45.247 | 57.833 | 50.464 | 69.849 | 6.565 | 6.677 | 10.115 | 10.369 | 9.068 | 9.359 |
| Minimum | -41.487 | 0.043 | -217.754 | 0.057 | -209.358 | 0.047 | -1.922 | 0.227 | -7.491 | 0.217 | -6.815 | 0.241 |
| 1st Quartile | 1.233 | 2.735 | -1.117 | 2.227 | -0.831 | 2.314 | 3.679 | 4.201 | 1.922 | 3.255 | 2.389 | 3.519 |
| Median | 4.427 | 6.001 | 2.666 | 6.404 | 2.911 | 6.363 | 5.996 | 6.543 | 5.160 | 6.467 | 5.372 | 6.531 |
| 3rd Quartile | 9.754 | 11.656 | 10.415 | 14.959 | 10.489 | 14.655 | 9.120 | 9.719 | 9.769 | 11.290 | 9.641 | 10.861 |
| Maximum | 227.310 | 249.270 | 458.364 | 521.416 | 428.839 | 538.489 | 48.157 | 49.007 | 87.847 | 91.660 | 85.278 | 88.653 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 6.907 | 7.176 | 6.916 | 7.589 | 6.910 | 7.466 | 6.886 | 7.015 | 6.879 | 7.200 | 6.889 | 7.152 |
| Equation (17)/(22) | 8.971 | 7.984 | 20.098 | 17.099 | 16.240 | 13.949 | 4.225 | 3.992 | 9.245 | 8.549 | 7.501 | 6.974 |
| var | 9.001 | 9.149 | 20.271 | 20.664 | 16.558 | 16.865 | 4.206 | 4.241 | 9.205 | 9.290 | 7.575 | 7.643 |
| Skewness | 0.916 | 0.925 | 1.248 | 1.270 | 1.137 | 1.156 | 0.615 | 0.617 | 0.820 | 0.825 | 0.767 | 0.772 |
| Kurtosis | 4.414 | 4.427 | 5.743 | 5.808 | 5.190 | 5.245 | 3.591 | 3.593 | 4.037 | 4.042 | 3.878 | 3.882 |
| Minimum | 0.162 | 0.406 | -1.882 | 0.327 | -1.062 | 0.364 | 1.473 | 1.617 | 0.203 | 0.443 | 0.326 | 0.613 |
| 1st Quartile | 4.758 | 5.004 | 3.666 | 4.310 | 3.981 | 4.494 | 5.426 | 5.546 | 4.696 | 4.999 | 4.906 | 5.156 |
| Median | 6.487 | 6.751 | 6.093 | 6.739 | 6.213 | 6.751 | 6.674 | 6.801 | 6.467 | 6.781 | 6.547 | 6.802 |
| 3rd Quartile | 8.574 | 8.852 | 9.264 | 9.949 | 9.050 | 9.623 | 8.135 | 8.269 | 8.616 | 8.947 | 8.497 | 8.768 |
| Maximum | 27.249 | 27.699 | 44.779 | 46.374 | 39.581 | 40.831 | 19.980 | 20.117 | 25.766 | 26.199 | 22.501 | 22.832 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 6.889 | 6.914 | 6.890 | 6.952 | 6.893 | 6.943 | 6.889 | 6.902 | 6.892 | 6.923 | 6.892 | 6.917 |
| Equation (17)/(22) | 0.807 | 0.798 | 1.736 | 1.710 | 1.415 | 1.395 | 0.401 | 0.399 | 0.861 | 0.855 | 0.702 | 0.697 |
| var | 0.710 | 0.710 | 1.618 | 1.619 | 1.309 | 1.310 | 0.352 | 0.352 | 0.815 | 0.815 | 0.654 | 0.655 |
| Skewness | 0.227 | 0.227 | 0.301 | 0.301 | 0.281 | 0.281 | 0.174 | 0.174 | 0.245 | 0.245 | 0.214 | 0.214 |
| Kurtosis | 3.106 | 3.106 | 3.148 | 3.148 | 3.114 | 3.114 | 3.041 | 3.041 | 3.089 | 3.089 | 3.059 | 3.059 |
| Minimum | 3.987 | 4.011 | 2.650 | 2.716 | 2.966 | 3.016 | 4.867 | 4.880 | 3.770 | 3.796 | 4.135 | 4.160 |
| 1st Quartile | 6.311 | 6.336 | 5.997 | 6.058 | 6.096 | 6.146 | 6.482 | 6.494 | 6.266 | 6.297 | 6.332 | 6.357 |
| Median | 6.855 | 6.879 | 6.828 | 6.891 | 6.837 | 6.887 | 6.871 | 6.883 | 6.857 | 6.888 | 6.866 | 6.891 |
| 3rd Quartile | 7.437 | 7.462 | 7.710 | 7.772 | 7.628 | 7.679 | 7.282 | 7.294 | 7.480 | 7.512 | 7.421 | 7.447 |
| Maximum | 10.770 | 10.797 | 13.431 | 13.498 | 13.226 | 13.273 | 9.556 | 9.569 | 10.954 | 10.983 | 10.449 | 10.476 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 5. Numerical Experiments -Uniform Distribution-.

| Case 1: $\sigma_{y jk} = 0.510$ | | | | | | | | | | | | |
|---------------------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Sample size | 10 | | | | | | 25 | | | | | |
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.513 | 0.510 | 0.545 | 0.510 | 0.515 | 0.510 | 0.511 | 0.510 | 0.521 | 0.510 | 0.511 |
| Equation (17)/(22) | 0.059 | 0.052 | 0.065 | 0.052 | 0.059 | 0.052 | 0.022 | 0.021 | 0.022 | 0.021 | 0.022 | 0.021 |
| var | 0.044 | 0.044 | 0.051 | 0.054 | 0.044 | 0.044 | 0.016 | 0.016 | 0.017 | 0.017 | 0.016 | 0.016 |
| Skewness | 0.596 | 0.596 | 0.762 | 0.722 | 0.597 | 0.596 | 0.328 | 0.328 | 0.337 | 0.324 | 0.328 | 0.328 |
| Kurtosis | 3.376 | 3.375 | 4.638 | 4.628 | 3.368 | 3.366 | 3.111 | 3.111 | 3.116 | 3.105 | 3.113 | 3.113 |
| Minimum | 0.033 | 0.034 | -0.066 | 0.030 | 0.031 | 0.036 | 0.098 | 0.098 | 0.089 | 0.093 | 0.097 | 0.099 |
| 1st Quartile | 0.357 | 0.360 | 0.346 | 0.377 | 0.357 | 0.362 | 0.421 | 0.422 | 0.419 | 0.430 | 0.421 | 0.423 |
| Median | 0.488 | 0.490 | 0.485 | 0.521 | 0.488 | 0.493 | 0.503 | 0.504 | 0.502 | 0.514 | 0.503 | 0.505 |
| 3rd Quartile | 0.641 | 0.643 | 0.646 | 0.687 | 0.640 | 0.646 | 0.591 | 0.592 | 0.593 | 0.605 | 0.591 | 0.593 |
| Maximum | 1.613 | 1.614 | 3.778 | 3.951 | 1.617 | 1.620 | 1.205 | 1.206 | 1.177 | 1.192 | 1.214 | 1.217 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.511 | 0.510 | 0.516 | 0.510 | 0.511 | 0.510 | 0.510 | 0.510 | 0.512 | 0.510 | 0.510 |
| Equation (17)/(22) | 0.011 | 0.010 | 0.011 | 0.010 | 0.011 | 0.010 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 |
| var | 0.008 | 0.008 | 0.008 | 0.008 | 0.008 | 0.008 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 | 0.004 |
| Skewness | 0.231 | 0.230 | 0.235 | 0.230 | 0.230 | 0.230 | 0.170 | 0.170 | 0.173 | 0.172 | 0.170 | 0.170 |
| Kurtosis | 3.061 | 3.061 | 3.071 | 3.068 | 3.061 | 3.061 | 3.006 | 3.006 | 3.010 | 3.009 | 3.006 | 3.006 |
| Minimum | 0.196 | 0.196 | 0.187 | 0.189 | 0.195 | 0.196 | 0.257 | 0.258 | 0.249 | 0.251 | 0.258 | 0.258 |
| 1st Quartile | 0.450 | 0.450 | 0.449 | 0.455 | 0.450 | 0.451 | 0.467 | 0.467 | 0.467 | 0.469 | 0.467 | 0.467 |
| Median | 0.506 | 0.507 | 0.506 | 0.512 | 0.506 | 0.507 | 0.508 | 0.508 | 0.508 | 0.510 | 0.508 | 0.508 |
| 3rd Quartile | 0.568 | 0.568 | 0.568 | 0.574 | 0.568 | 0.568 | 0.551 | 0.551 | 0.551 | 0.554 | 0.551 | 0.551 |
| Maximum | 0.932 | 0.933 | 0.934 | 0.941 | 0.932 | 0.933 | 0.798 | 0.798 | 0.793 | 0.796 | 0.798 | 0.798 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.510 | 0.510 | 0.511 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 | 0.510 |
| Equation (17)/(22) | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| var | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| Skewness | 0.140 | 0.140 | 0.141 | 0.141 | 0.141 | 0.141 | 0.087 | 0.087 | 0.088 | 0.088 | 0.087 | 0.087 |
| Kurtosis | 3.025 | 3.025 | 3.028 | 3.028 | 3.025 | 3.025 | 3.063 | 3.063 | 3.063 | 3.063 | 3.063 | 3.063 |
| Minimum | 0.389 | 0.389 | 0.389 | 0.389 | 0.389 | 0.389 | 0.412 | 0.412 | 0.412 | 0.412 | 0.412 | 0.412 |
| 1st Quartile | 0.488 | 0.488 | 0.488 | 0.488 | 0.488 | 0.488 | 0.494 | 0.494 | 0.494 | 0.495 | 0.494 | 0.494 |
| Median | 0.509 | 0.509 | 0.509 | 0.510 | 0.509 | 0.509 | 0.509 | 0.509 | 0.509 | 0.510 | 0.509 | 0.510 |
| 3rd Quartile | 0.532 | 0.532 | 0.532 | 0.532 | 0.532 | 0.532 | 0.525 | 0.525 | 0.525 | 0.525 | 0.525 | 0.525 |
| Maximum | 0.664 | 0.664 | 0.665 | 0.666 | 0.664 | 0.664 | 0.616 | 0.616 | 0.616 | 0.616 | 0.616 | 0.616 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 5. Numerical Experiments -Uniform Distribution-.

| Case 2: $\sigma_{y X} = 0.510$ | | | | | | | | | | | | |
|--------------------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Sample size | 10 | | | | | | 25 | | | | | |
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.509 | 0.599 | 0.509 | 0.647 | 0.508 | 0.731 | 0.510 | 0.540 | 0.510 | 0.555 | 0.510 | 0.578 |
| Equation (17)/(22) | 0.073 | 0.052 | 0.107 | 0.052 | 0.145 | 0.052 | 0.023 | 0.021 | 0.024 | 0.021 | 0.025 | 0.021 |
| var | 0.038 | 0.051 | 0.058 | 0.078 | 0.083 | 0.116 | 0.010 | 0.011 | 0.011 | 0.013 | 0.012 | 0.014 |
| Skewness | 0.714 | 0.813 | 2.428 | 2.593 | 3.456 | 3.531 | 0.168 | 0.165 | 0.386 | 0.361 | 0.445 | 0.379 |
| Kurtosis | 6.253 | 6.574 | 41.824 | 39.565 | 90.701 | 58.000 | 3.052 | 3.044 | 3.768 | 3.702 | 3.988 | 3.767 |
| Minimum | -1.852 | 0.003 | -3.606 | 0.003 | -7.368 | 0.004 | 0.153 | 0.162 | 0.154 | 0.169 | 0.153 | 0.173 |
| 1st Quartile | 0.371 | 0.439 | 0.354 | 0.462 | 0.341 | 0.514 | 0.440 | 0.466 | 0.437 | 0.476 | 0.437 | 0.496 |
| Median | 0.494 | 0.582 | 0.482 | 0.618 | 0.474 | 0.690 | 0.507 | 0.537 | 0.506 | 0.550 | 0.505 | 0.574 |
| 3rd Quartile | 0.629 | 0.739 | 0.628 | 0.788 | 0.629 | 0.887 | 0.576 | 0.610 | 0.577 | 0.627 | 0.577 | 0.654 |
| Maximum | 2.946 | 3.659 | 8.443 | 9.805 | 11.662 | 12.334 | 1.056 | 1.096 | 1.396 | 1.458 | 1.366 | 1.476 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.524 | 0.510 | 0.531 | 0.510 | 0.542 | 0.510 | 0.517 | 0.510 | 0.520 | 0.510 | 0.526 |
| Equation (17)/(22) | 0.011 | 0.010 | 0.011 | 0.010 | 0.011 | 0.010 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 |
| var | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 | 0.002 |
| Skewness | 0.106 | 0.106 | 0.168 | 0.163 | 0.177 | 0.165 | 0.068 | 0.068 | 0.084 | 0.083 | 0.085 | 0.083 |
| Kurtosis | 3.006 | 3.005 | 3.117 | 3.107 | 3.129 | 3.105 | 3.018 | 3.018 | 3.024 | 3.025 | 3.024 | 3.025 |
| Minimum | 0.252 | 0.262 | 0.257 | 0.268 | 0.255 | 0.273 | 0.314 | 0.318 | 0.317 | 0.323 | 0.317 | 0.326 |
| 1st Quartile | 0.464 | 0.476 | 0.462 | 0.482 | 0.462 | 0.492 | 0.478 | 0.485 | 0.478 | 0.488 | 0.478 | 0.493 |
| Median | 0.509 | 0.523 | 0.509 | 0.530 | 0.508 | 0.540 | 0.509 | 0.516 | 0.509 | 0.520 | 0.509 | 0.525 |
| 3rd Quartile | 0.555 | 0.571 | 0.555 | 0.578 | 0.556 | 0.590 | 0.541 | 0.549 | 0.542 | 0.552 | 0.542 | 0.558 |
| Maximum | 0.825 | 0.849 | 0.942 | 0.969 | 0.942 | 0.981 | 0.709 | 0.720 | 0.757 | 0.776 | 0.765 | 0.791 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.510 | 0.511 | 0.510 | 0.512 | 0.510 | 0.513 | 0.510 | 0.511 | 0.510 | 0.511 | 0.510 | 0.511 |
| Equation (17)/(22) | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| var | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 | 0.001 |
| Skewness | 0.139 | 0.139 | 0.140 | 0.140 | 0.140 | 0.140 | 0.086 | 0.085 | 0.086 | 0.086 | 0.086 | 0.086 |
| Kurtosis | 3.034 | 3.033 | 3.036 | 3.036 | 3.036 | 3.036 | 3.025 | 3.025 | 3.024 | 3.024 | 3.024 | 3.024 |
| Minimum | 0.373 | 0.374 | 0.373 | 0.374 | 0.373 | 0.375 | 0.420 | 0.420 | 0.419 | 0.420 | 0.419 | 0.421 |
| 1st Quartile | 0.488 | 0.489 | 0.488 | 0.490 | 0.488 | 0.491 | 0.494 | 0.495 | 0.494 | 0.495 | 0.494 | 0.496 |
| Median | 0.509 | 0.511 | 0.509 | 0.511 | 0.509 | 0.512 | 0.510 | 0.510 | 0.510 | 0.511 | 0.510 | 0.511 |
| 3rd Quartile | 0.531 | 0.533 | 0.531 | 0.533 | 0.531 | 0.535 | 0.525 | 0.526 | 0.525 | 0.526 | 0.525 | 0.527 |
| Maximum | 0.643 | 0.645 | 0.645 | 0.647 | 0.644 | 0.648 | 0.613 | 0.613 | 0.612 | 0.614 | 0.612 | 0.614 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 5. Numerical Experiments -Uniform Distribution-.

| Case 3: $\sigma_{y jk} = 1.118$ | | | | | | | | | | | | |
|---------------------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Sample size | 10 | | | | | | 25 | | | | | |
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.115 | 1.225 | 1.114 | 1.352 | 1.115 | 1.389 | 1.117 | 1.153 | 1.118 | 1.194 | 1.117 | 1.200 |
| Equation (17)/(22) | 0.397 | 0.289 | 0.596 | 0.331 | 0.566 | 0.308 | 0.128 | 0.115 | 0.154 | 0.132 | 0.141 | 0.123 |
| var | 0.328 | 0.354 | 0.468 | 0.559 | 0.456 | 0.518 | 0.102 | 0.105 | 0.126 | 0.135 | 0.115 | 0.120 |
| Skewness | 1.505 | 1.502 | 2.207 | 2.202 | 2.133 | 2.265 | 0.570 | 0.557 | 0.850 | 0.815 | 0.724 | 0.687 |
| Kurtosis | 14.641 | 15.279 | 16.858 | 16.258 | 19.287 | 21.102 | 3.562 | 3.549 | 4.391 | 4.313 | 4.023 | 3.963 |
| Minimum | -0.946 | 0.046 | -6.136 | 0.047 | -3.885 | 0.050 | 0.179 | 0.188 | 0.177 | 0.190 | 0.177 | 0.197 |
| 1st Quartile | 0.702 | 0.799 | 0.658 | 0.849 | 0.660 | 0.905 | 0.889 | 0.923 | 0.866 | 0.934 | 0.876 | 0.956 |
| Median | 1.025 | 1.138 | 0.978 | 1.216 | 0.994 | 1.272 | 1.089 | 1.125 | 1.071 | 1.149 | 1.080 | 1.165 |
| 3rd Quartile | 1.425 | 1.549 | 1.409 | 1.689 | 1.428 | 1.725 | 1.312 | 1.351 | 1.321 | 1.405 | 1.316 | 1.406 |
| Maximum | 15.633 | 16.584 | 13.776 | 14.530 | 16.626 | 18.175 | 3.128 | 3.184 | 3.789 | 4.023 | 3.433 | 3.531 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.119 | 1.136 | 1.119 | 1.155 | 1.119 | 1.158 | 1.117 | 1.126 | 1.117 | 1.135 | 1.117 | 1.136 |
| Equation (17)/(22) | 0.061 | 0.058 | 0.071 | 0.066 | 0.065 | 0.062 | 0.030 | 0.029 | 0.034 | 0.033 | 0.032 | 0.031 |
| var | 0.048 | 0.049 | 0.058 | 0.060 | 0.053 | 0.054 | 0.023 | 0.023 | 0.028 | 0.028 | 0.025 | 0.026 |
| Skewness | 0.396 | 0.392 | 0.580 | 0.566 | 0.500 | 0.486 | 0.258 | 0.256 | 0.375 | 0.370 | 0.323 | 0.319 |
| Kurtosis | 3.257 | 3.254 | 3.624 | 3.606 | 3.473 | 3.462 | 3.100 | 3.099 | 3.233 | 3.227 | 3.182 | 3.179 |
| Minimum | 0.385 | 0.396 | 0.374 | 0.391 | 0.386 | 0.409 | 0.601 | 0.609 | 0.569 | 0.581 | 0.581 | 0.596 |
| 1st Quartile | 0.965 | 0.981 | 0.949 | 0.982 | 0.956 | 0.995 | 1.011 | 1.019 | 1.000 | 1.017 | 1.006 | 1.024 |
| Median | 1.105 | 1.122 | 1.097 | 1.133 | 1.100 | 1.140 | 1.111 | 1.119 | 1.107 | 1.125 | 1.109 | 1.128 |
| 3rd Quartile | 1.257 | 1.275 | 1.265 | 1.304 | 1.262 | 1.303 | 1.216 | 1.225 | 1.222 | 1.241 | 1.220 | 1.239 |
| Maximum | 2.289 | 2.307 | 2.573 | 2.628 | 2.436 | 2.477 | 1.930 | 1.941 | 1.912 | 1.935 | 1.954 | 1.977 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.118 | 1.120 | 1.118 | 1.121 | 1.118 | 1.122 | 1.118 | 1.119 | 1.118 | 1.120 | 1.118 | 1.120 |
| Equation (17)/(22) | 0.006 | 0.006 | 0.007 | 0.007 | 0.006 | 0.006 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 |
| var | 0.006 | 0.006 | 0.007 | 0.007 | 0.006 | 0.006 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 | 0.003 |
| Skewness | 0.162 | 0.162 | 0.209 | 0.209 | 0.185 | 0.184 | 0.108 | 0.108 | 0.134 | 0.134 | 0.117 | 0.117 |
| Kurtosis | 3.013 | 3.013 | 3.077 | 3.077 | 3.041 | 3.041 | 3.048 | 3.048 | 3.062 | 3.061 | 3.052 | 3.052 |
| Minimum | 0.842 | 0.844 | 0.832 | 0.835 | 0.831 | 0.834 | 0.908 | 0.908 | 0.889 | 0.891 | 0.909 | 0.911 |
| 1st Quartile | 1.065 | 1.067 | 1.062 | 1.065 | 1.063 | 1.067 | 1.081 | 1.082 | 1.079 | 1.081 | 1.080 | 1.082 |
| Median | 1.116 | 1.118 | 1.115 | 1.119 | 1.115 | 1.119 | 1.117 | 1.118 | 1.117 | 1.119 | 1.117 | 1.119 |
| 3rd Quartile | 1.169 | 1.170 | 1.172 | 1.175 | 1.170 | 1.174 | 1.153 | 1.154 | 1.156 | 1.158 | 1.154 | 1.156 |
| Maximum | 1.513 | 1.514 | 1.533 | 1.537 | 1.542 | 1.546 | 1.378 | 1.379 | 1.366 | 1.368 | 1.370 | 1.372 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 5. Numerical Experiments -Uniform Distribution-.

Case 4: $\sigma_{y|jk} = 1.000$

| Sample size | 10 | | | | | | 25 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.001 | 1.091 | 1.000 | 1.206 | 1.000 | 1.223 | 0.999 | 1.029 | 0.998 | 1.064 | 0.999 | 1.067 |
| Equation (17)/(22) | 0.316 | 0.232 | 0.481 | 0.270 | 0.436 | 0.247 | 0.102 | 0.093 | 0.126 | 0.108 | 0.113 | 0.099 |
| var | 0.264 | 0.282 | 0.379 | 0.452 | 0.362 | 0.405 | 0.083 | 0.085 | 0.104 | 0.111 | 0.093 | 0.097 |
| Skewness | 1.174 | 1.164 | 2.124 | 2.164 | 2.103 | 2.265 | 0.590 | 0.580 | 0.899 | 0.865 | 0.745 | 0.712 |
| Kurtosis | 5.852 | 5.955 | 15.041 | 16.253 | 17.794 | 20.338 | 3.690 | 3.674 | 4.593 | 4.522 | 4.306 | 4.241 |
| Minimum | -0.584 | 0.047 | -3.721 | 0.046 | -2.829 | 0.054 | 0.180 | 0.190 | 0.180 | 0.190 | 0.180 | 0.199 |
| 1st Quartile | 0.631 | 0.709 | 0.585 | 0.750 | 0.594 | 0.794 | 0.795 | 0.823 | 0.768 | 0.827 | 0.781 | 0.847 |
| Median | 0.920 | 1.011 | 0.872 | 1.081 | 0.890 | 1.118 | 0.973 | 1.003 | 0.956 | 1.022 | 0.967 | 1.036 |
| 3rd Quartile | 1.278 | 1.378 | 1.271 | 1.515 | 1.278 | 1.521 | 1.176 | 1.207 | 1.178 | 1.252 | 1.177 | 1.250 |
| Maximum | 5.910 | 6.216 | 10.743 | 13.037 | 13.002 | 14.269 | 2.977 | 3.034 | 3.134 | 3.304 | 4.450 | 4.548 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 0.999 | 1.013 | 0.999 | 1.030 | 0.999 | 1.030 | 1.000 | 1.007 | 1.000 | 1.016 | 1.000 | 1.016 |
| Equation (17)/(22) | 0.049 | 0.046 | 0.058 | 0.054 | 0.052 | 0.049 | 0.024 | 0.023 | 0.028 | 0.027 | 0.025 | 0.025 |
| var | 0.039 | 0.040 | 0.048 | 0.049 | 0.043 | 0.044 | 0.019 | 0.019 | 0.023 | 0.024 | 0.021 | 0.021 |
| Skewness | 0.410 | 0.406 | 0.575 | 0.563 | 0.489 | 0.478 | 0.269 | 0.267 | 0.383 | 0.379 | 0.323 | 0.319 |
| Kurtosis | 3.272 | 3.271 | 3.546 | 3.528 | 3.433 | 3.424 | 3.136 | 3.136 | 3.260 | 3.256 | 3.195 | 3.193 |
| Minimum | 0.411 | 0.419 | 0.398 | 0.417 | 0.399 | 0.423 | 0.534 | 0.540 | 0.506 | 0.517 | 0.516 | 0.529 |
| 1st Quartile | 0.859 | 0.872 | 0.844 | 0.873 | 0.853 | 0.884 | 0.904 | 0.911 | 0.893 | 0.908 | 0.899 | 0.914 |
| Median | 0.986 | 1.000 | 0.978 | 1.009 | 0.981 | 1.014 | 0.994 | 1.001 | 0.991 | 1.006 | 0.993 | 1.008 |
| 3rd Quartile | 1.125 | 1.139 | 1.133 | 1.167 | 1.127 | 1.160 | 1.090 | 1.097 | 1.097 | 1.113 | 1.093 | 1.108 |
| Maximum | 2.107 | 2.124 | 2.383 | 2.423 | 2.206 | 2.258 | 1.660 | 1.669 | 1.796 | 1.815 | 1.792 | 1.809 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 1.000 | 1.001 | 1.000 | 1.003 | 1.000 | 1.003 | 1.000 | 1.001 | 1.000 | 1.001 | 1.000 | 1.001 |
| Equation (17)/(22) | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.002 | 0.002 | 0.003 | 0.003 | 0.002 | 0.002 |
| var | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.005 | 0.002 | 0.002 | 0.003 | 0.003 | 0.002 | 0.002 |
| Skewness | 0.167 | 0.166 | 0.208 | 0.207 | 0.182 | 0.182 | 0.117 | 0.117 | 0.141 | 0.141 | 0.132 | 0.132 |
| Kurtosis | 3.075 | 3.075 | 3.094 | 3.094 | 3.079 | 3.079 | 3.071 | 3.071 | 3.063 | 3.063 | 3.072 | 3.072 |
| Minimum | 0.759 | 0.760 | 0.732 | 0.734 | 0.745 | 0.748 | 0.826 | 0.826 | 0.809 | 0.810 | 0.810 | 0.811 |
| 1st Quartile | 0.953 | 0.954 | 0.949 | 0.952 | 0.952 | 0.955 | 0.967 | 0.968 | 0.964 | 0.966 | 0.966 | 0.967 |
| Median | 0.998 | 1.000 | 0.997 | 1.000 | 0.998 | 1.001 | 0.999 | 1.000 | 0.999 | 1.000 | 0.999 | 1.000 |
| 3rd Quartile | 1.045 | 1.046 | 1.048 | 1.051 | 1.046 | 1.049 | 1.032 | 1.033 | 1.034 | 1.036 | 1.033 | 1.034 |
| Maximum | 1.318 | 1.319 | 1.351 | 1.355 | 1.321 | 1.324 | 1.214 | 1.215 | 1.243 | 1.245 | 1.234 | 1.236 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.

Table 5. Numerical Experiments -Uniform Distribution-.

| Case 5: $\sigma_{y x} = 6.890$ | | | | | | | | | | | | |
|--------------------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Sample size | 10 | | | | | | 25 | | | | | |
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 6.886 | 8.826 | 6.914 | 11.437 | 6.948 | 11.304 | 6.887 | 7.460 | 6.871 | 8.269 | 6.880 | 8.064 |
| Equation (17)/(22) | 89.284 | 39.921 | 273.079 | 85.493 | 255.709 | 69.743 | 20.461 | 15.968 | 48.386 | 34.197 | 39.013 | 27.897 |
| var | 73.274 | 76.123 | 189.405 | 194.614 | 188.836 | 194.565 | 17.568 | 17.847 | 41.738 | 42.406 | 34.538 | 35.049 |
| Skewness | 2.759 | 3.012 | 2.518 | 3.008 | 3.165 | 3.946 | 1.092 | 1.105 | 1.493 | 1.515 | 1.374 | 1.389 |
| Kurtosis | 23.382 | 25.464 | 16.877 | 18.783 | 31.337 | 46.078 | 4.870 | 4.905 | 6.484 | 6.576 | 5.769 | 5.820 |
| Minimum | -29.946 | 0.020 | -204.962 | 0.044 | -135.808 | 0.085 | -1.136 | 0.232 | -3.933 | 0.172 | -3.418 | 0.181 |
| 1st Quartile | 1.244 | 2.876 | -1.183 | 2.255 | -0.891 | 2.354 | 3.833 | 4.380 | 2.098 | 3.428 | 2.524 | 3.670 |
| Median | 4.734 | 6.445 | 2.855 | 6.685 | 3.075 | 6.710 | 6.159 | 6.723 | 5.340 | 6.715 | 5.569 | 6.724 |
| 3rd Quartile | 10.096 | 11.981 | 11.158 | 15.365 | 11.081 | 15.100 | 9.150 | 9.727 | 9.972 | 11.356 | 9.786 | 11.001 |
| Maximum | 161.940 | 167.812 | 179.531 | 203.065 | 370.455 | 460.304 | 39.672 | 40.183 | 78.119 | 80.845 | 53.620 | 56.525 |

| Sample size | 50 | | | | | | 100 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 6.893 | 7.158 | 6.908 | 7.563 | 6.912 | 7.453 | 6.900 | 7.028 | 6.896 | 7.214 | 6.896 | 7.155 |
| Equation (17)/(22) | 8.971 | 7.984 | 20.098 | 17.099 | 16.240 | 13.949 | 4.225 | 3.992 | 9.245 | 8.549 | 7.501 | 6.974 |
| var | 7.738 | 7.796 | 18.267 | 18.403 | 14.919 | 15.022 | 3.722 | 3.737 | 8.604 | 8.641 | 6.948 | 6.975 |
| Skewness | 0.737 | 0.739 | 1.012 | 1.016 | 0.936 | 0.937 | 0.512 | 0.513 | 0.738 | 0.739 | 0.664 | 0.664 |
| Kurtosis | 3.858 | 3.859 | 4.484 | 4.491 | 4.262 | 4.260 | 3.383 | 3.383 | 3.820 | 3.821 | 3.638 | 3.638 |
| Minimum | 0.201 | 0.411 | -0.649 | 0.356 | -0.319 | 0.466 | 1.036 | 1.161 | 0.209 | 0.505 | 0.285 | 0.491 |
| 1st Quartile | 4.886 | 5.147 | 3.765 | 4.407 | 4.079 | 4.608 | 5.525 | 5.652 | 4.789 | 5.100 | 4.999 | 5.256 |
| Median | 6.547 | 6.814 | 6.195 | 6.837 | 6.314 | 6.855 | 6.738 | 6.865 | 6.537 | 6.855 | 6.613 | 6.870 |
| 3rd Quartile | 8.548 | 8.820 | 9.300 | 9.963 | 9.105 | 9.660 | 8.090 | 8.221 | 8.629 | 8.950 | 8.475 | 8.737 |
| Maximum | 29.321 | 29.573 | 36.865 | 37.688 | 32.626 | 33.335 | 17.138 | 17.308 | 24.775 | 25.115 | 21.457 | 21.745 |

| Sample size | 500 | | | | | | 1000 | | | | | |
|--------------------|----------|------------|----------|------------|----------------|------------|----------|------------|----------|------------|----------------|------------|
| Variables | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | | Z_1 | | Z_2 | | $\{Z_1, Z_2\}$ | |
| Estimator | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent | unbiased | consistent |
| Estimates | 6.887 | 6.912 | 6.898 | 6.961 | 6.895 | 6.945 | 6.890 | 6.903 | 6.893 | 6.924 | 6.892 | 6.917 |
| Equation (17)/(22) | 0.807 | 0.798 | 1.736 | 1.710 | 1.415 | 1.395 | 0.401 | 0.399 | 0.861 | 0.855 | 0.702 | 0.697 |
| var | 0.817 | 0.819 | 1.748 | 1.751 | 1.429 | 1.432 | 0.400 | 0.400 | 0.856 | 0.857 | 0.700 | 0.701 |
| Skewness | 0.279 | 0.280 | 0.358 | 0.359 | 0.336 | 0.336 | 0.181 | 0.181 | 0.242 | 0.242 | 0.238 | 0.238 |
| Kurtosis | 3.143 | 3.143 | 3.191 | 3.191 | 3.189 | 3.189 | 3.068 | 3.068 | 3.100 | 3.100 | 3.078 | 3.078 |
| Minimum | 3.589 | 3.610 | 2.812 | 2.870 | 3.026 | 3.077 | 4.681 | 4.692 | 3.533 | 3.563 | 3.975 | 3.999 |
| 1st Quartile | 6.264 | 6.289 | 5.969 | 6.032 | 6.054 | 6.103 | 6.452 | 6.464 | 6.253 | 6.285 | 6.310 | 6.335 |
| Median | 6.842 | 6.867 | 6.822 | 6.882 | 6.833 | 6.882 | 6.872 | 6.884 | 6.855 | 6.886 | 6.858 | 6.883 |
| 3rd Quartile | 7.471 | 7.497 | 7.744 | 7.807 | 7.662 | 7.712 | 7.308 | 7.321 | 7.495 | 7.527 | 7.437 | 7.463 |
| Maximum | 11.430 | 11.455 | 13.691 | 13.760 | 13.156 | 13.217 | 10.122 | 10.135 | 11.468 | 11.504 | 10.926 | 10.956 |

Unbiased: unbiased estimator; Consist: consistent estimator; Estimates: the sample mean from 50000 estimated causal effects on the variance; (17)/(22): the exact and asymptotic variances derived from equations (17) and (22) with Table 3; Var: empirical variances from 50000 estimated causal effects on the variance.