FIRST- AND SECOND-GRADE PROSPECTIVE TEACHERS CONSTRUCTING COLLECTIVE DEFINITIONS OF POLYGON DIAGONALS

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*The purpose of this study was to enhance first- and second-grade prospective teachers’ knowledge of the concept of polygon diagonals. A total of 23 prospective teachers participated in the study, and data were collected from pre- and post-questionnaires and observations. The pre-questionnaire’s findings indicated that all participants provided incorrect definitions. However, the post-questionnaire findings showed that most (87%) participants provided the correct definition. The observations also revealed that participants struggled to connect formal definitions to non-prototype examples of diagonals in the initial learning process. However, engaging in the analysis of mathematical events helped participants reconstruct the definition of polygon diagonals and identify the critical attributes of this concept.*

**INTRODUCTION**

Early-grades teachers are responsible for laying the foundation for future learning in mathematics; research has consistently shown that teachers who have a strong understanding of the mathematical concepts they teach are better able to support their students in developing their own understanding and skills (e.g., Sherstha, 2022). Specifically, knowledge of geometry and teachers' geometric thinking level affect their students' geometric thinking levels (Pavlovičová et al., 2022). Despite this importance, teachers' knowledge of geometry is limited; Tsamir et al. (2014) reported that only a small percentage of all early-years teachers in their study defined geometric concepts in a minimal way. Shahbari's (2022) study revealed a low level of knowledge in geometry compared to the other mathematics fields among practicing and prospective first- and second-grade mathematics teachers. Therefore, there is a need to develop prospective teachers' knowledge. Understanding mathematical definitions of concepts is essential to identifying critical features of geometric shapes and developing geometrical understanding (Haj-Yahya et al., 2022). A useful tool for helping teachers better understand mathematical ideas is engaging in mathematical events analyses (Stockero et al., 2019). Analyses of mathematical events allow for the creation of a community of learners and the opportunity for discussion and argumentation around mathematical concepts. The process of argumentation, in which claims are presented, evaluated, and either accepted or rejected, is seen as a way to build up a collective understanding (Toulmin, 2003).

The current study examined how participants' understanding of polygon diagonals evolved into a more collective understanding as they participated in the analyses of mathematical events related to the definition of polygon diagonals and engaged in discussion and argumentation with their peers. This may have included examining how participants' definitions of polygon diagonals changed and how they used evidence and reasoning to support their ideas.

**THEORETICAL BACKGROUND**

Acquiring mathematical concepts needs two main components based on the model proposed by Vinner & Hershkowitz (1980): the concept definition and the concept image. The concept definition is the verbal-mathematical description of the concept, while the concept image is the cognitive structure that represents the concept in the learner's mind. When the concept image matches the concept definition, the concept is learned. However, a mismatch between these two components can negatively impact students' ability to identify examples, construct examples, and engage in proving processes (Marchis, 2012). Mathematical definitions are essential for understanding the meanings of mathematical concepts and for solving problems such as constructing theorems and proofs (e.g., Haj-Yahya et al., 2022). However, numerous studies have identified difficulties that both students and teachers face when defining general mathematical concepts and understanding the structure and meaning of these definitions (e.g., Haj-Yahya, 2021). For example, Haj-Yahya et al. (2019) found that teachers struggled with using "uneconomical definitions", "missing definitions", or rejecting equivalent definitions of geometrical concepts. In the current study, we focused on polygon diagonals, which has been identified as an essential but difficult concept. Previous research has found that students often struggle with understanding polygon diagonals. For example, Wilson and Schmidt (2005) also found that high school students had misconceptions about polygon diagonals, such as the belief that the number of diagonals equals the number of sides in the polygon.

Vinner (1991) suggests using activities that present learners with a conflict that can be resolved through a precise mathematical definition. This helps students understand the precision and importance of definitions as a tool for effective mathematical communication. The current study adopts mathematical event analyses to answer Vinner’s recommendation. Mathematical events are cases that occur in the mathematics classroom and refer to problems or tasks in mathematics that students engage with. The teacher plays an important role in these mathematical events, as they respond to the problem presented by the students and guide them toward a solution (Tirosh, 2019). It is important to monitor the mathematical progress of a class at the collective level rather than focusing on the individual thinking of each participant. This allows for an understanding of the accepted mathematical meanings within a class community when the class is treated as its own entity (Toulmin, 2003).

Research questions: (1) Are first- and second-grade prospective teachers familiar with the definition of polygon diagonals? (2) How do the personal understandings of prospective teachers in the definition of polygon diagonals affect the collective definitions constructed by the whole group through their engagement in mathematical events analysis related to the definition?

**METHOD**

Research participants and context

The present study was conducted with 23 prospective teachers who were studying for their teaching certification for first and second grades at a college for teacher training in the Arab community in Israel. The course content focused on four areas of geometric thinking: properties of shapes, place and space relations, transformations and symmetry, and visualization. The research focused on two meetings that emphasized the concept of diagonals. The teaching and learning process throughout the course, including the two meetings, was based on discussions in mathematical events, which emphasized different ways of thinking and common errors in the definitions of chosen subjects.

Data sources and analyses

The data for this study were collected from three sources: 1) pre- and post-questionnaires, which included a definition task and tasks that examined pedagogical knowledge; 2) observations of class discussions, which were recorded by video and transcribed word for word. The pre- and post-questionnaires’ data were analyzed using thematic content analysis based on categories identified in Tsamir et al.'s (2015) research on the definition of geometric concepts. Each definition was evaluated based on the dimensions listed in Table 1. The correct definition is according to the Ministry of Education's website (https://retro.education.gov.il/tochniyot\_limudim/math/metzolaim.htm#cm6). The correct definition of a polygon diagonal is “a line segment that connects any two non-adjacent vertices.” This is considered a minimal definition. We started with documenting the observation using Toulmin’s model (Toulmin, 2003) by creating an argumentation log. We then constructed an arguments core that consisted of three parts: data, claim, and warrant. More parts would be added to these parts according to the participants’ responses, such as backings, qualifiers, and rebuttals. After that, we applied Stephen and Rasmussen’s (2008) method for determining when a classroom community’s ways of reasoning or mathematical ideas become normative by looking across class sessions to see functions as-if-shared (FAIS).

**FINDINGS**

Diagonal definition: before and after event analysis

In Table 1 below, we present the diagonal polygon definitions that emerged from the pre- and post-questionnaire and show representative examples of the participants' definitions.

Table 1: Correct and incorrect polygon diagonal definitions

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | Frequency | Examples: a polygon diagonal is… |
|  |  | Pre-questionnaire | Post-questionnaire |
| Correct definitions | Minimal | - | 13 (57%) | * A line segment that connects any two non-adjacent vertices
 |
| Non-minimal | - | 7 (30%) | * A line segment that connects any two non-adjacent vertices. The diagonal is completely external or completely internal, or partly internal and partly external
 |
| Incorrect definitions | Insufficient (missing critical attribute/s) | 13 (57%) | 2 (9%) | * A line segment inside the polygon
* A line segment that connects two vertices
* A line segment that connects a vertex to a parallel vertex
* A straight line that connects a vertex to a parallel vertex
 |
| Based on non-critical attribute/s | 10 (43%) | 1 (4%) | * A straight line that divides a shape into two equal parts
* Crosses the polygon
 |

We can see from Table 1 above that all the prospective teachers provided incorrect definitions in the pre-questionnaire. 57% of the participants wrote an incorrect, insufficient definition that was missing critical attributes when the vast majority mentioned only the critical attribute of “a line segment” or “straight line” without mentioning the non-adjacent vertex. 43% of the participants added non-critical attributes in the diagonal concept definition, such as using the attributes “Inside the polygon”, “crosses the polygon”, and “divided to two equal parts”. These non-critical attributes indicated a limited concept image of a diagonal being just inside the polygon.

Furthermore, the findings of the post-questionnaire show that most participants improved their correct definitions. Only 13% of the participants gave incorrect definitions. 87% of the participants wrote the correct definition (minimal or non-minimal), and 57% of the participants gave a minimal definition that included necessary and sufficient attributes, critical attributes of “a line segment” in addition to “non-adjacent vertices”. 30% of the participants gave a non-minimal definition which includes attributes focused on the diagonal location targeted to expand the concept image such as “Completely or partly internal”, these additional attributes indicate that their concept image of a diagonal was changed.

Definition development and the relationship with a concept image

The collective discussions and participation in events analysis based on non-prototype examples of diagonals contributed to understanding the concept definition of diagonals. The event constructed based on prototype examples initially caused various arguments shown in Table 2 related to diagonal definition. The arguments that emerged from prospective teachers show that although the diagonal definition was presented in front of them throughout the meeting, they were not always aware of the gap between the intuitive example of the diagonal and the analytical aspect arising from the definition. However, during the argumentative discourse, the prospective teachers identified which critical attributes the diagonal has, and which others should not be considered. The evidence for this finding is the last argument that came up in the meeting which refers to the need to check all the critical attributes found in the diagonal definition. Such a process is called the relationship between concept image and its definition.

Table 2: Arguments that emerged during participants’ mathematical events analysis

|  |  |
| --- | --- |
| NO. | Arguments title |
| 1 | The diagonal definition is incomplete |
| 2 | The diagonal definition is complete (counterargument to argument 1) |
| 3 | The line segment that connects two vertices in a polygon that is entirely outside the polygon is not a diagonal |
| 4 | The number of diagonals in a concave quadrilateral is two |
| 5 | The number of diagonals adjacent vertices in a concave quadrilateral |
| 6 | The locations of all diagonals in a concave octagon that connect the vertices to the other vertices in the polygon |
| 7 | The line segment that connects a vertex to a side finder is not a diagonal |
| 8 | The line segment that connects two vertices and is entirely contained in the polygon is a diagonal |
| 9 |  The line segment that passes through a side is not a diagonal |
| 10 | The line segment that passes through a side is a diagonal (counterargument to argument 9) |
| 11 | The line segment that goes partially inside the polygon and partially outside it is called a diagonal |
| 12 | The line segment direction that connects two adjacent vertices in a polygon is not a critical attribute of diagonal  |

Due to space limitations, one episode was selected. Selected rows from the chosen episode are displayed in Episode 1 below:

Episode 1: The diagonals number in a concave square

1 Instructor: How many diagonals does the polygon in front of you have?

2 Sina: There is another diagonal in the middle. From vertex 1 to vertex 4

3 Instructor: So, how many diagonals does a polygon have?

4 Riwaa: There is only one diagonal

5 Instructor: Why?

6 Riwaa: According to the definition?

7 Instructor: What did you infer from the definition?

8 Riwaa: The only diagonal is the one connecting vertex 1 and vertex 4

24 Instructor: What are the numbers of adjacent vertices in the polygon?

25 Riwaa: 2 and 4…..4 and 3

26 Instructor: Are these just the adjacent vertices?

27 All participants: No

28 Instructor: Riwaa, please…

29 Riwaa: Ahhh…..2 and 4, 4 and 3, 2 and 1, 1 and 3

30 Instructor: What do you conclude about vertices 1 and 4? Are they adjacent vertices?

31 Riwaa: No. are not adjacent

32 Riwaa: Are not adjacent. I can connect a diagonal between them

33 Riwaa: Are not adjacent. I can connect a diagonal between them

34 Instructor: Are you convinced that the red segment is a diagonal?

35 Riwaa: Now yes…Like Sijal said. There are also external diagonals

The discussion began with a question: How many diagonals does the polygon in front of you have? [1]. The claim was made by Riwaa [4]. In terms of Toulman’s Model, Riwaa’s claim can be broken down as follows:

Argument 4: The number of diagonals in a concave quadrilateral is two [2-8]

Claim 1: There is only one diagonal [4]

Data 1: [2]

Warrant 1: According to the definition [6]

Warrant 2: The only diagonal is the one connecting vertex 1 and vertex 4 [8]

Later, the next argument about the adjacent vertex number was made by the same teacher, Riwaa:

Argument 5: The number of diagonals adjacent vertices in a concave quadrilateral [1-33]

Claim 1: Vertices 2 and 4, 4 and 3 are adjacent [25]

Data 1: [1]

Warrant 1: Vertices 2 and 1, 1 and 3 are adjacent [29]

Warrant 2: Vertices 1 and 4 are not adjacent [31]

Warrant 3: Vertices 3 and 2 are not adjacent. I can connect a diagonal between them [33]

Based on the first argument, Riwaa's understanding of the concept image of a polygon diagonal did not align with its definition. Despite reading and understanding the definition, Riwaa was unable to identify a diagonal outside of the polygon and eliminated that example from the collection of correct examples for the concept. However, during the discussion, specifically in the second argument, it became clear that Riwaa did not have a clear understanding of the concept of "adjacent vertices". As a result, she excluded the external diagonal from all diagonals of the displayed polygon. This was evident when, after learning about adjacent vertices, she understood the definition well, specifically the critical attributes of diagonal definition. After Argument 5, several participants agreed with Riwaa's understanding [33]. This broad consensus is a sign of normative agreement and strengthens the participants’ perspectives in the context of Argument 6. Therefore, it can be concluded that this provides indirect evidence of the idea that "if the vertices are not adjacent, you can connect a diagonal between them" became an idea that functions as-if-shared; in other words, becoming a shared understanding among the participants.

**DISCUSSION**

The main objective of this study is to examine the knowledge of prospective first- and second-grade teachers regarding the definition of diagonals in polygons, and how this knowledge is reconstructed. The study's results show that before the intervention, a majority of the prospective teachers provided incorrect definitions in the pre-questionnaire, which is consistent with previous research (e.g. Haj-Yahya, 2021; Tsamir et al., 2014). However, there was a significant improvement in the post-questionnaire, with more participants providing correct definitions (see Table 1). Furthermore, the analysis of polygon diagonal meetings revealed the teachers' methods of reasoning and shared ideas. These shared ideas were organized around the properties of polygon diagonal lines, which constitute the collective mathematical growth of the classroom community in understanding the concept of diagonals according to Toulmin (2003) model analyses. In the pre-questionnaire, about 40% of the participants included non-critical attributes incorrectly, however, this tendency dropped drastically in the post-questionnaire. Instead, more participants included non-minimal correct attributes in their definitions, which could expand the understanding of the diagonal concept by including attributes exclusive to non-prototypical examples of the concept, such as being external to the polygon (Vinner & Hershkowitz, 1980). These results align with the claim about the interaction between the concept definition and the concept image (Vinner, 1991), and the arguments that emerged during the mathematical events analysis strengthen the quantitative results (see Table 2). Given these findings, it is recommended that future research focus on analyzing mathematical events related to the definition of other geometric concepts.

References

Haj-Yahya, A. (2021). Students' conceptions of the definitions of congruent and similar triangles. *International Journal of Mathematical Education in Science and Technology*, 1-25.‏

Haj-Yahya, A., Daher, W., & Swidan, O. (2019). In-service teachers' conceptions of parallelogram definitions. In *Eleventh Congress of the European Society for Research in Mathematics Education* (No. 12). Freudenthal Group; Freudenthal Institute; ERME.‏

Haj-Yahya, A., Hershkowitz, R., & Dreyfus, T. (2022). Investigating students' geometrical proofs through the lens of students definitions. *Mathematics Education Research Journal (MERJ)*. 1-27.

Marchis, I. (2012). Preservice primary school teachers' elementary geometry knowledge. *Acta Didactica Napocensia*, *5*(2), 33-40.‏

Shahbari, J. A. (2022). Investigation of mathematical-pedagogical knowledge among prospective teachers in the early childhood program at the college for Arabic speakers. In A. Rasslan, & D. Hasidov (Eds.), *Special issues in early childhood mathematics education research* (pp 246-265). Leiden, The Netherlands: Brill.

Shrestha, R. (2022). Teachersʼ Content Knowledge and Pedagogical Content Knowledge for Teaching: As Preconditions to Develop Studentsʼ Mathematical Thinking at Grade 1-3 in Nepal. *NUE Journal of International Educational Cooperation, 15*, 123-132.

Stockero, S. L., Leatham, K. R., Ochieng, M. A., Zoest, L. R., & Peterson, B. E. (2019).Teachers’ orientations toward using student mathematical thinking as a resource during whole‑class discussion*. Journal of Mathematics Teacher Education*, 1-31.

Tirosh, D., Tsamir, P., Levenson, E. S., & Barkai, R. (2019). Using theories and research to analyze a case: learning about example use. *Journal of Mathematics Teacher Education*, *22*(2), 205-225.

Toulmin, S. (2003). *The Uses of Argument* (2nd ed.). Cambridge: Cambridge University Press. doi:10.1017/CBO9780511840005

Tsamir, P., Tirosh, D., Levenson, E., Barkai, R. & Tabach, M., (2014). Early-years teachers' concept image and concept definition: triangles, circles and cylinders. *ZDM mathematics Education, 47*(3), 1-13.

Vinner, S., & Hershkowitz, R. (1980). Concept image and common cognitive paths in the development of some simple geometrical concepts. In R. Karplus (Ed.), *Proceedings of the Conference of the International Group for Psychology of Mathematics Education* (pp. 177-184). University of California, Berkeley, California.

Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht: Kluwer Academic Publishers.

Wilson, J.C., & Schmidt, S.L. (2005). Diagnosing and Remedying Students' Misconceptions about Polygon Diagonals. Journal for Research in Mathematics Education, 36(4), 270-303.