**Introduction**

Integro-differential equations appeared very naturally in various applications (for example, see [9, 10, 17, 18, 25]), which explains the interest in the theory related to these equations (for example, see [2, 3]) such systems are found in models for many mechanical systems.

Consider the non-linear system of integro-differential equations:

Where

 is n-dimensional vector

are constant matrices

 is a symmetric positive-definite matrix

Let's define the matrices and as

 Where are symmetric and are skew symmetric matrices. We will use terminology commonly found in mechanics to classify the forces acting in the system:

1. Potential force -
2. Dissipative force -
3. Gyroscopic force -
4. Bounded damping forces -
5. Non-linear force -

If the non-linear force is absent, the stability and oscillation of the system (1) are well studied. Lyapunov's direct method has been successfully employed to investigate the stability of ODE systems when the linear approximation is non-critical.

As for the non-linear force , we have the following result:

**Theorem**

 Let the system

Where

 are vector-polynomials of degree in .

Be non-resonant, i.e. For any integer-valued vector

.

The zero solution of (0.1a) is Birkhoff stable (stable in any finite nonlinear approximation)

If only potential and non-linear forces present in (1) it is reducible to

Where

The spectrum of linear approximation has n pairs of pure imaginary eigenvalues. Such a system is used substantially in the theory of nonlinear oscillations, and it achieves critical stability.

In this paper we want to investigate the stability of an integro-differential system with non-linear force when the entire spectrum or part of its linear approximation is on the imaginary axis.

First, we need to reduce the integro-differential system to the corresponding ODE system. This can be found in the first part of this article. The idea of reducing to a system of ordinary differential equations when studying stability was presented in [5]. In the second part of the article, we will present the method and give definitions related to the method for reducing nonlinear systems to the simplest form. This method will be called normalization. After this method has been applied to a system, the system will be said to be a system reduced to normal form or simply "n.f."

In part 3 of the article, we will find the stability/instability condition for the zero solution of a system reduced to normal form.

Then, in part 4, we will show how the integral addend can be used as a way to control stability, which means that if the nonlinear oscillator has an unstable solution, then it is possible to choose coefficients for the integral addend that will make the solution stable or vice versa.

**Part 1 - Reduction method**

The idea and development of the reduction method for the intergo-differential equation were described in works by Domoshnitsky and Goltser [5, 6].

We will present how a system of integro-differential equations can be reduced to a system of ODEs.

Let us consider the system of integro-differential equation:

Let us assume that

Then the system (1.1) can be written in the form

If the kernel is a square-integrable function, then in a Hilbert space it can be represented in the form:

Where

 are matrices continuous on . Additionally, we assume that are invertible matrices. We can write in the following form

And we will assume that

We will write

 Where in the form (1.4)

Let us introduce a new variable

Where

,

Then the system (6) can be reduced to the ODE system

Where

The question is, under what conditions it is possible to obtain a matrix with constant coefficients? We will find the conditions that are sufficient and necessary to transform the matrix to a matrix with constant coefficients.

Let be the fundamental system of solutions and let us assum we can write in the following form

Where

 – Lyapunov matrix

 – constant matrix

We substitute (1.5) into the system

 and get

**Theorem**

 The system of equations

can be reduced to a system with constant coefficients

if

This condition is necessary and sufficient if we use Erugin's results on reducible systems.

**[Erugin's] Theorem**

The linear differential system

is reducible if and only if some fundamental matrix can be represented in the form of a Lyapunov matrix multiplied by the exponential of the product of the independent variable and the constant matrix , i.e.

Now let us consider a few cases:

(a) is a Cauchy function, so we write

thus is Cauchy functional equation:

Because is a Cauchy function, it has the following properties

Where

 is the order of the linear differential operator , which depends on the structure of .

Thus, from system (1.1), we derived the following 2n-dimensional system of ODEs:

What to do if is not a Cauchy function? For this, let's consider the following case

(b) As we know from Leontief [], a rather broad class of functions can be expanded in the form of a general Dirichlet series. Let the functions have the following representation:

Let us consider systems whose kernel is the difference.

Assume that is an analytical function and can be written as

Then can be expanded to a series (1.6) of exponents (see Leontief)

From (1.6), we learn

1. There is the possibility of a countable system of ODEs.
2. Each exponent is a Cauchy function of some ODE.
3. The equations are different, depending on whether is real or complex.

There is another way to decompose : for example, as a Fourier series of the following form

We can substitute

and obtain the desired decomposition.

(c) The periodic case.

Let us assume that is a periodic matrix with period . From the equation

Using Floquet theory, we conclude that in this case the kernel contains the matrix

Where has period with respect to .

Then, using Floquet's theorem as a special case of the theorem on the system's reducibility, the matrix can be reduced to a system with constant coefficients.

In this work, we want to use a kernel that has a periodic representation.

Example

As a first example, let us consider the application of the reduction method to an analysis of a non-linear oscillator with a non-linear force represented as an integral.

Let's consider system (1), where , ,

So the system will look like this

 And it is easy to see that the operator has the form

 and is a second-order operator.

After applying the reduction method, the system looks this

**Part 2 - Normal Form**

Let us briefly present the idea of reducing the system to normal form. For this, we will use terminology that can be found in Bibikov

 Consider two formal systems of ordinary differential equations

and

Where

 are formal power series

Definition

We say that systems (2.1) and (2.2) are formally equivalent if there exists a change of variables

Where

is a formal power series, which reduces (2.1) to (2.2)

Let be a vector whose coordinates are eigenvalues of matrix .

Theorem

 If

 Then system (2.1) is formally equivalent to any system (2.2) and is uniquely defined in (2.3).

We seek the simplest form of such a system. It is convenient to assume that is in Jordan canonical form. This can be achieved through linear–singular changes of variables.

 Accordingly, we consider the system:

Definition

 When considering system (2.5), we say that coefficients of any power series corresponding to a pair that satisfies

are resonant and the corresponding term is called a resonant term. But if

is true, then we say that the coefficient and corresponding term are non-resonant. Equation (2.6) is called a resonance equation.

Definition

 When all non-resonant terms are equal to zero, the system (2.5) is said to be in normal form (NF).

As an example of finding the normal form, consider a system of nonlinear oscillators where, for example, the first oscillator is perturbed by a force represented by the integral:

Using the reduction method described above, we obtain the following system of equations

Let's make the following replacement

Then system (2.8) can be written in matrix form as follows:

Where

Let's perform the change of variables to bring matrix to diagonal form

We substitute this change into (2.9) and obtain

We multiply the last equation on the left by and get

After introducing the new notation, we obtain

where

is a new non-linearity

We make the substitution

After the substitution, we want to get

Substituting (2.11) into (2.10), we obtain

We use the expression (2.12)

After simplifying the last expression, we get

Where

is obtained after substituting a series into a series. Equation (2.13) is called a homological equation.

Now we need to investigate when equation (2.13) has a solution for .

Let us consider the equation and the equation for finding the coefficients of the terms of the -th-order form

And we get

 The last condition will help us find all the resonance values of vector . Our result here can be expressed in the following theorem.

**Theorem**

 The system of equations

Using the substitution

leads to this form

Where

 has only resonant terms that can be found using the formula

Example of normal form

We use condition (2.8) in a system with two pairs of imaginary eigenvalues in order to find the resonance terms that remain after the normalization method

Then the first nonlinear term will be and

Then the first nonlinear term will be and

As we saw above, if there is no internal resonance, then the second-order terms do not affect the structure of the normal form, so we can choose them to be equal to zero. Then the nonlinearities can be chosen as follows

Let's start the normalization after reducing the integro-differential equation to a system of ODEs. So we'll consider the following system of equations

We make the change of variables (2.15)

And we get the following ODE system

The last system can be written as follows

Using more concise notation, the last system can be written as follows

The eigenvalues of matrix will be

After transforming the linear part of the system (2.16) to diagonal form using linear transform , , where the transform matrix and its inverse transform are:

We substitute this change into (2.16) and obtain

We multiply the last equation on the left by and get

After introducing the new notation, we obtain

where

 is a new nonlinearity obtained after substituting a series into a series

Let's make a substitution that is close to the identity

After the substitution, we want to get

The coefficients, which need to be found in normal form, will be:

We will write out only the final result for the coefficients of the normal form

The normal form up to the third order has the following structure

**Theorem**

 After normalization, the following system of equations

 is reducible to

and the coefficients of the normal form are found in the equality (2.17)

**Part 3 - Analysis of the stability of the zero solution of the equation ( )**

The system (2.18) was considered in works by Lyapunov, Malkin, Veretennikov, Molchanov, and Goltser. We use some of the results obtained in these works to analyze the stability of the zero solution of this system.

Let us consider the following system

We make the substitution

And get

Equating the real and imaginary parts, we get

Alternatively, we can write

The following theorem can provide an answer regarding the stability of the solution to the latter system

**Theorem** [Goltser]

1. To achieve the asymptotic stability of the solution of the system

 regardless of the terms , it is necessary and sufficient for and one the following two conditions to be satisfied :

1. If the solution for the system ( ) is asymptotically stable, regardless of the terms , then for the system ( ) there exists a Lyapunov function , , with a sign-definite derivative because of the system ( ).

 Applying the last theorem to our example, we obtain

Or

For the zero solution to be A-stable, the following conditions must be met

or

Analyzing the conditions (), we obtain

**Theorem**

 For the zero solution of the system

to be A-stable, the following inequalities must hold

Comment

The last inequality gives the condition for the coefficient so that the solution to the equation with an integral addend is stable, but this means that and if this is true, the solution of the equation without the integral addend will be stable

**Part 4 - using integral addend as a way to control the stability of a non-linear oscillator.**

Let the following equation be given

such that and the zero solution are unstable.

Using an integral addend in the following form

we want to achieve the stability of the zero solution of the new equation

After using the reduction principle, we obtain the following system of equations

Let's make a change of coordinates and get the following system of equations

The structure of the normal form will be the same, and as before, we will write out only the resulting coefficients of the normal form.

Where

We make a change of coordinates and get the following system

Using the theorem (), we need to verify the following inequalities

For convenience, we introduce new notation

Then we get

The last inequality can be replaced by the following inequality

In each inequality, we find an expression for

Let us assume that

Then we can write

Or

Therefore, if we choose such that the last inequality () holds, then we can choose so that inequality () is also satisfied.

**Theorem**

 The zero solution to the equation

will be A-stable if the following conditions are met

**References**

[1] Ju.N. Bibikov, Local Theory of Nonlinear Analytic ODEs, in: Lecture Notes in Mathematics, vol. 702, Springer-Verlag, 1979

[2] C. Corduneanu, Integral Equations and Stability Feedback Systems. New York, London: Academic Press, 1973.

[3] C. Corduneanu, Integral Equations and Applications, Cambridge University Press, Cambridge, 1991

[4] A.D. Brjuno, The Local Method of Nonlinear Analysis, Nauka, Moscow, 1972 (in Russian)

[5] A. Domoshnitsky. Exponential stability of convolution integro-differential equations, Functional-Differential Equations, 1998, vol. 5, pp. 445-455.

[6] A. Domoshnitsky, Ya. Goltser, One approach to study of stability of IDE, Nonlinear Analysis 47 (2001) 3885-3896.

[7] A. Domoshnitsky, Ya. Goltser, Hopf bifurcations of IDEs, Qualitative Theory of Differential Equations 3 (2000) 1-11.

[8] A.Domoshnitsky, Ya. Goltser, Floquet theorem and stability of linear integro-differential equations, Functional Differential Equations 10(3-4) (2003) 463-471.

[9] A.D. Drozdov and V.B. Kolmanovskii, Stability in Viscoelasticity, North Holland, Amsterdam, 1994.

[10] M. Fabrizio and A. Morro. Mathematical Problems in Linear Viscoelasticity, SIAM Stud.Appl.Math., Philadelphia, 1992.

[11] F.R. Gantmacher, Theory of Matrices, Nauka. 1967.

[12] Ya. Goltser, The process of normalization and solutions of bifurcation problems of the oscillation and stability theory: A synopsis, Functional Differential Equations 1 (1993) 108-125.

[13] Ya. Goltser, On the stability of differential equation systems with the spectrum on the imaginary axis, Functional Differential Equations 4(1-2) (1997) 47-63.

[14] Ya. Goltser, A. Domoshnitsky, Bifurcation and stability of IDEs, Nonlinear Analysis 47(2001) 953-967.

[15] Ya. Golster, A.L. Kunitsyn, On the stability of an autonomous system with internal resonance, Prikladnay Matematika I Mekhanika 59(6) (1075) 974-985 (in Russian). English trans.: Journal of Applied Mathematics and Mechanics, PMM, 39(6)(1976).

[16] Ya.Goltser, E.Litsyn, Volterra integro-differential equations and infinite systems of ODEs, Mathematical and Computer Modeling 42(2005) 221-233.

[17] J.M. Golden and G.A. Graham. Boundary value problems in linear viscoelasticity. Springer-Verlag, 1998.

[18] M.E. Gurtin and A.C. Pipkin, A General Theory of Heat conduction with Finite Wave Speeds, Archive for Rational Mechanics and Analysis, 31(1968) 113-

[19] G.V. Kamenkov, selected works, vol. 1, 1979; vol.2, 1972.

[20] A.M. Lyapunov, Collected works, vol. 2, Moscow, 1956.

[21] I.G. Malkin, Theory of Movement Stability, Nauka, 1966 (in Russian). Translation by the Atomic Energy Association, AGE-TR.-3352.

[22] I.G. Malkin, Nekotorye zadachi nelinejnykh kolebanij, 2004

[23] D.R. Merkin, Introduction to the Theory of Movement Stability, Nauka, 1971 (in Russian)

[24] A.M. Molchanov, Stability and Oscillations of Nonlinear approximation, Doklady Akademii Nauk SSSR 141 (1967) 24-27.

[25] A. Novick-Cohen. Conserved Phase-Field Equations with Memory, in Curvature Flows and Related topics, A. Damlamian, J. Spruck and A. Vistin, eds., pp. 179-197, GAKUTO Internat.Ser. Math. Sci. Appl., 5. Gakkotosho, Tokyo, 1995.

[26] V.G. Veretnikov, Stability and Oscillations of nonlinear Systems, Nauka, Moscow, 1984 (in Russian).

[27] V.R. Rumyantsev, A.S. Oziraner, Stability and Motion Stabilization for part of variables, Nauka, Moscow, 1987 (in Russian).