**BEN-GURION UNIVERSITY OF THE NEGEV**

**FACULTY OF ENGINEERING SCIENCES**

**SCHOOL OF ELECTRICAL AND COMPUTER ENGINEERING**

**DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING**



**ANALYSIS OF LINEAR TIME-VARYING & PERIODIC SYSTEMS**

THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE MSc. DEGREE

By: Oren Fivel

Supervised by:

Professor Izchak Lewkowicz

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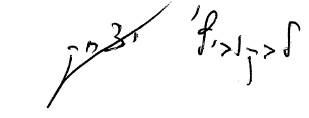
By: Oren Fivel

Supervised by:

Professor Izchak Lewkowicz



Author: Oren Fivel ……………… Date: 26-Oct-2020



Supervisor: Professor Izchak Lewkowicz Date: 26-Oct-2020

Chairman of Graduate Studies Committee:

Name: ………………………… ……………… Date: ………

# Abstract

This thesis applies *Floquet theory* to analyze linear periodic time-varying (LPTV) systems, which are represented by a system of ordinary differential equations (ODEs) that depend on a time variable and have a matrix of coefficients with period (or equivalently, with frequency ).

According to *Floquet theory*, the transition matrix of a LPTV system represented by a square periodic-function matrix can be expressed as the product of a square periodic function matrix and an exponentiated square matrix of the form , where is a constant matrix (independent of ). Despite the validity of *Floquet theory*, it is difficult to find an analytical closed form for the matrices and when the transition matrix is unknown. In essence, it is difficult to find an analytical solution for a LPTV system (i.e., a closed form for its transition matrix).

The objective of this work is to determine how the frequency of a LPTV system affects the solutions (i.e., a stability analysis) and also to characterize a sufficiently large family of periodic matrices with a finite number of harmonics such that the periodic part of the solution also has a finite number of harmonics. In addition, for each family, an effective procedure is required to find the matrices and .

The research method involves studying periodic matrices with frequency as a free parameter (i.e., not a fixed number) to obtain a sufficiently large family of LPTV systems so that their transition matrices are defined by the frequency . In this work, we focus on cases in which the Fourier coefficients of are finite polynomials in , is a finite polynomial in , and the Fourier coefficients of do not depend on .

The research results show that for a given family of periodic matrices we can compare the powers of that multiply the harmonics (i.e., is part of the coefficients multiplying the cosine [sine] factors in even [odd] representations or of the exponential factors in complex representations) to determine the matrices and . In addition, the results lead to relations between LPTV systems at frequency and the associated linear time-invariant system, which is defined by having zero frequency (). The significance of using the frequency as a free parameter is that is allows the stability of the LPTV system to be determined based on how depends on frequency.

***Keywords:*** Floquet theory; Linear ordinary differential equations; LTV systems, Periodic systems; Fourier series; Matrix equation; Eigen decomposition; Frequency-domain analysis; Power-coefficient comparison; Finite number of harmonics

# Thesis Overview

This thesis analyzes linear periodic time-varying (LPTV) systems, which are defined by linear ordinary differential equations (ODEs) with a matrix of periodic functions , in turn defined by a frequency parameter . This work investigates how the frequency parameter ω affects LPTV systems and their solutions. This work is organized as follows:

1. ‎**CHAPTER 1** introduces the topic, discusses the background of LPTV systems and *Floquet theory*, summarizes previous work by other authors, and suggests how to analyze and solve a LPTV system based on its frequency , which is a free parameter (i.e., not fixed).
2. ‎**CHAPTER 2** gives details of *Floquet theory* and discusses properties that emanate from this theory.
3. ‎**CHAPTER 3** presents examples of LPTV systems and solutions with finite and infinite harmonics and outlines how the frequency affects the solution.
4. ‎**CHAPTER 4** implements the cosine-sine Fourier transform of LPTV systems that converts the LPTV system’s ODE into a matrix algebraic equation that is similar to an eigen decomposition problem.
5. ‎**CHAPTER 5** outlines our suggestion for comparing powers of to solve LPTV systems, which is incorporated with the information obtained from the associated linear time-invariant (LTI) system defined by using in the LPTV system.
6. ‎**CHAPTER 6** summarizes this work, discusses its scientific contributions, and presents avenues for future research.

The appendixes provide additional information, such as mathematical background, an overview of previous work, and suggestions for future work. The appendices in this work are organized as follows:

1. ‎**APPENDIX A** outlines the notion of Fourier series for matrices (exponential form and cosine-sine form) and properties thereof.
2. ‎**APPENDIX B** applies exponential Fourier series analysis to LPTV systems and compares this approach with the cosine-sine form described in ‎CHAPTER 4.
3. ‎**APPENDIX C**, Sections ‎C.1 and ‎C.2 outline complex LPTV systems and how they can be converted to real LPTV systems based on the isomorphism . In addition, Section ‎C.2 outlines how to decompose the state vector into even and odd parts and, furthermore, how to rewrite the LPTV system matrix using even and odd matrix blocks.
4. ‎**APPENDIX D** outlines the main results of (Wang, 2017), which generalize the notion eigen decomposition of linear time-varying systems (this is a continuation of the literature survey of Section ‎1.4.2).
5. ‎**APPENDIX E** outlines the main results of (Wereley, 1991), which suggest implementing LPTV systems by using generalized tools to analyze and control LTI systems (e.g., transfer functions and zeros and poles in the and domains). This is a continuation of the literature survey of Section ‎1.4.5.

# Acknowledgments

I would like to thank my supervisor Professor Izchak Lewkowicz, with whom I worked for two intensive years. Professor Lewkowicz has been a supportive tutor and has helped me tremendously during my research. I thank him for teaching me to think outside the box, to be persistent, and to strive to perfect my performance. I also thank him for improving my learning skills and for sharing outstanding ideas with me throughout my studies.

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# Table of Contents

[Abstract i](#_Toc54342273)

[Thesis Overview iii](#_Toc54342274)

[Acknowledgments v](#_Toc54342275)

[Table of Contents vi](#_Toc54342276)

[Figure List viii](#_Toc54342277)

[Table List viii](#_Toc54342278)

[Acronyms and Abbreviations x](#_Toc54342279)

[CHAPTER 1. Introduction 1](#_Toc54342280)

[1.1 Background 1](#_Toc54342281)

[1.2 Linear Time-Varying Differential Equations 2](#_Toc54342282)

[1.3 Linear Periodic Time Varying System 6](#_Toc54342283)

[1.4 Literature Survey: Previous Research and Applications 10](#_Toc54342284)

[1.4.1 Stability Analysis of LPTV System 10](#_Toc54342285)

[1.4.2 Dynamic Eigen Decomposition of LTV System 13](#_Toc54342286)

[1.4.3 Small-Perturbation Approach 13](#_Toc54342287)

[1.4.4 Factorization of Alternative Transition Matrix of LPTV System 14](#_Toc54342288)

[1.4.5 Generalization of LTI-System Tools for LPTV Systems 14](#_Toc54342289)

[1.5 Suggested Approach and Scope of Work 16](#_Toc54342290)

[CHAPTER 2. Floquet Theory and its Properties 18](#_Toc54342291)

[2.1 Overview 18](#_Toc54342292)

[2.2 Floquet Theory 18](#_Toc54342293)

[2.3 Relation between LPTV and LTI Systems 21](#_Toc54342294)

[2.4 Zero Average LPTV System Matrix 22](#_Toc54342295)

[2.5 Nonunique Decomposition of LPTV Transition Matrix 23](#_Toc54342296)

[2.5.1 Similarity Transformation of](#_Toc54342297) *[R](#_Toc54342297)* [23](#_Toc54342297)

[2.5.2 Similarity Transformation of](#_Toc54342298) *[A](#_Toc54342298)*[(](#_Toc54342298)*[t](#_Toc54342298)*[) 23](#_Toc54342298)

[2.6 Floquet Theory: Zero-Trace System Matrix 25](#_Toc54342299)

[2.7 Summary 26](#_Toc54342300)

[CHAPTER 3. Examples of Analyses of LPTV Systems 28](#_Toc54342301)

[3.1 Overview 28](#_Toc54342302)

[3.2 Number of Harmonics in LPTV Systems and in the Periodic Parts of their Transition Matrices 29](#_Toc54342303)

[3.2.1 General 29](#_Toc54342304)

[3.2.2 Examples 30](#_Toc54342305)

[3.3 Notions of Frequency ω as a Free Parameter in LPTV Systems 33](#_Toc54342306)

[3.3.1 General 33](#_Toc54342307)

[3.3.2 Examples 34](#_Toc54342308)

[3.4 Summary 40](#_Toc54342309)

[CHAPTER 4. Fourier Analysis of LPTV Systems 41](#_Toc54342310)

[4.1 Overview 41](#_Toc54342311)

[4.2 Cosine-Sine Fourier Analysis of LPTV Systems 41](#_Toc54342312)

[4.2.1 General 41](#_Toc54342313)

[4.2.2 Even-Odd Decomposition 42](#_Toc54342314)

[4.3 Summary 46](#_Toc54342315)

[CHAPTER 5. LPTV Systems with Coefficients as Polynomials in ω 48](#_Toc54342316)

[5.1 Overview 48](#_Toc54342317)

[5.2 Cosine and Sine Fourier Coefficients as Polynomials in ω 48](#_Toc54342318)

[5.3 Relation to LTI Systems 52](#_Toc54342319)

[5.4 Transformation of LPTV Matrix to Canonical Form at ω = 0 56](#_Toc54342320)

[5.5 Examples 58](#_Toc54342321)

[5.6 Summary 78](#_Toc54342322)

[CHAPTER 6. Discussion 79](#_Toc54342323)

[6.1 Contribution of the Research 79](#_Toc54342324)

[6.2 Suggestions for Future Research 80](#_Toc54342325)

[APPENDIX A. Fourier Series for Matrices 83](#_Toc54342326)

[APPENDIX B. Exponential Fourier Analysis for LPTV Systems 88](#_Toc54342327)

[B.1 General 88](#_Toc54342328)

[B.2 Real-Imaginary Decomposition 90](#_Toc54342329)

[APPENDIX C. Representing LPTV Systems by 2×2 Real Blocks and Split Complex Numbers 94](#_Toc54342330)

[C.1 Representing Complex LPTV System by 2×2 Real Blocks 94](#_Toc54342331)

[C.2 Representing Even-Odd Decomposition of LPTV Systems by 2×2 Blocks 96](#_Toc54342332)

[APPENDIX D. Dynamic Eigen Decomposition of LTV Systems 99](#_Toc54342333)

[APPENDIX E. Generalization of LTI-System Tools for LPTV Systems 102](#_Toc54342334)

[References 107](#_Toc54342335)

**Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.**

# Figure List

**[Figure ‎3‑1](#_Toc54342336)** [Eigenvalue of as functions of frequency (Markus & Yamabe, 1960). (a) λ](#_Toc54342336)[1](#_Toc54342336)[; (b) λ](#_Toc54342336)[2](#_Toc54342336)[. 38](#_Toc54342336)

**Error! Hyperlink reference not valid.**

# Table List

**[Table ‎3‑1](#_Toc54342337)** [Examples of 2×2 LPTV Systems. Case 1: L is infinite, p is infinite. 30](#_Toc54342337)

**[Table ‎3‑2](#_Toc54342338)** [Examples of 2×2 LPTV Systems. Case 2: L is infinite, p is finite. 30](#_Toc54342338)

**[Table ‎3‑3](#_Toc54342339)** [Examples of 2×2 LPTV Systems. Case 3: L is finite, p is infinite. 30](#_Toc54342339)

**[Table ‎3‑4](#_Toc54342340)** [Examples of 2×2 LPTV Systems. Case 4: L is finite, p is finite. 32](#_Toc54342340)

**[Table ‎3‑5](#_Toc54342341)** [Stability status of e](#_Toc54342341)[tR](#_Toc54342341) [for different values of (Markus & Yamabe, 1960). 38](#_Toc54342341)

**Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.Error! Hyperlink reference not valid.**

# Acronyms and Abbreviations

LHS left-hand side

LPTV linear periodic time varying

LTI linear time invariant

LTV linear time varying

ODE ordinary differential equation

RHS right-hand side

# Introduction

## Background

This thesis applies *Floquet theory* (Floquet, 1883) to solve linear periodic time-varying (LPTV) systems, which may be represented by systems of ordinary differential equations (ODEs). A LPTV system is defined by a square periodic matrix with period [i.e., ]. According to *Floquet theory* (see **Theorem ‎2.1** below for full details), the solution of a LPTV system defined by the matrix includes the product of the matrices and , where is a constant (independent of ). *Floquet theory* is setup by the following equation:

|  |  |
| --- | --- |
| (‎1.1) |  |

The product is called the transition matrix of the matrix of the LPTV system and is a special case of the transition matrix of a linear time-varying (LTV) system [see Eqs. (‎1.3) and (‎1.4) for the definition of a transition matrix].

The notion of a transition matrix of a LPTV system might appear straightfoward enough because its structure is known. However, in practice, it is difficult to find an analytical closed-form solution of a LPTV system, or, equivalently, for the matrices and . Moreover, no procedure is available in the literature for solving the ODE of a general LPTV system (or even for specific families thereof), or, equivalently, for finding the matrices and when the transition matrix is unknown.

## Linear Time-Varying Differential Equations

Let be the time domain of the independent variable (), with the initial time denoted . Suppose (called the *state vector* or simply the *state*) is a solution to the following LTV system of differential equations:

|  |  |
| --- | --- |
| (‎1.2) |  |

where is some matrix (called the *system matrix*) that is independent of the variable (i.e., the system is linear). The solution is then represented as

|  |  |
| --- | --- |
| (‎1.3) |  |

where is the initial time, is the initial position for , and is the *transition matrix* of that solves the following differential equation:

|  |  |
| --- | --- |
| (‎1.4) |  |

where is the identity matrix, and . Hereinafter, to simplify notation, we denote functions of as elements of real matrices; e.g., is denoted , is denoted , is denoted .

To analyze a LTV system, we consider shifting the diagonal of the matrix of the LTV system by some scalar function (Lewkowicz, 1999). Suppose is some known scalar function. By shifting the diagonal of the square matrix to produce a new system matrix , then this new system matrix has the transition matrix:

|  |  |
| --- | --- |
| (‎1.5) |  |

Refer to (Rugh, 1996) Property 4.2, if a LTV system matrix and, then the transition matrix is obtained by exponentiating this anti-derivative (similar to the scalar ODE case), i.e.,

|  |  |
| --- | --- |
| (‎1.6) |  |

Two special cases of Eq (‎1.6) allow for solutions:

1. If is a constant matrix, then . This case is referred to as a linear time-invariant (LTI) system.
2. If is a diagonal matrix [i.e., ], then is also a diagonal matrix such that .

Case 2 is a special case of Eq (‎1.6), because both and are diagonal square matrices and, thus, trivially commute. Case 1 is a special case of Eq (‎1.6) because and , so and trivially commute because any matrix commutes with itself and with a scalar multiple of itself. LTI systems are easy to solve because we can compute based on the eigen composition of . However, it is very difficult to find a transition matrix for a general LTV system represented by a matrix . Moreover, it is not always valid to exponentiate the antiderivative of to achieve the transition matrix. In other words, in general.

**Example ‎1.1:** **Cauchy–Euler Equation**

Consider the following LTV system matrix:

|  |  |
| --- | --- |
| (‎1.7) |  |

where , . Based on a solution in which is a power of ], the transition matrix obtained is

|  |  |
| --- | --- |
| (‎1.8) |  |

where

|  |  |
| --- | --- |
| (‎1.9) |  |

After multiplying out , we arrive at:

|  |  |
| --- | --- |
| (‎1.10) |  |

In contrast, is given by:

|  |  |
| --- | --- |
| (‎1.11) |  |

so that .□.

A (deterministic) *state space* of a LTV system is defined by the matrix set such that:

|  |  |
| --- | --- |
| (‎1.12) |  |

where (with suitable dimensions) is the state vector, is the input vector (usually the control input), and is the output vector (system measurements). The solution for and its corresponding output are:

|  |  |
| --- | --- |
| (‎1.13) |  |

where the term is the homogeneous part of the solution [i.e., the solution to Eq. (‎1.2), ], and the term is the forced part of the solution [with zero initial condition, ]. In contrast with LTI systems, solving a LTV system with an impact impulse input (i.e., a Dirac delta function) is difficult because of the time dependence of the LTV system. Therefore, to solve a general LTV system, we must find from Eq. (‎1.4).

An LTI system can be generated by a local linearization of an autonomous nonlinear system adjacent to equilibrium points. In other words, although an autonomous nonlinear system is difficult to solve directly, several initial conditions and initial times can be set to shift the solution without deforming it. On the other hand, linearizing a general nonlinear system adjacent to equilibrium points would create a LTV system, which remains difficult to solve. Moreover, it is impossible to set several initial conditions with several initial times because the system is time dependent (changing the initial time would change the solution).

In this work, we focus on a periodic system matrix so that for some [ is called a -periodic function of ]. Despite *Floquet theory* giving the structure of the transition matrix of a LPTV system, it remains difficult to compute the exact components of the transition matrix of this LPTV system.

## Linear Periodic Time-Varying System

**Definition ‎1.2**: a LPTV system is defined as a linear system of ODEs (namely, a LTV system) that can be represented by a coefficient matrix of -periodic functions, i.e.,

|  |  |
| --- | --- |
| (‎1.14) |  |

□

LPTV systems are used to describe the motion of lunar perigee(Hill, 1886),(Hill, 1878), vibrations of stretched elliptical membranes (Mathieu, 1868), the motion of side rods of a locomotive (Meissner, 1918), elliptical waveguides (McLachlan, 1947),(Pillay & Kumar, 2017)**,** the motion of gravitationally stabilized Earth-pointing satellites (Schechter, 1964), quadrupole mass spectrometry (Dawson, 1976), the rolling motion of ships (Jovanoski & Robinson, 2009), micromechanical tuning fork gyroscope dynamics (King, 1989), pendulum dynamics (Seyranian & Seyranian, 2006), helicopter rotors (Friedmann, 1986), wind turbines (Stol, Balas, & Bir, 2002), multistage DC-DC converters (Li, Guo, Ren, Zhang, & Zhang, 2017), etc. A LPTV system can be obtained by linearizing a nonlinear system that has periodic components (e.g., the vertical motion of a pendulum undergoes periodic motion, where the pendulum’s angle with the vertical axis and its angular velocity is the output).

#### **Example ‎1.3: Hill Equation**

The *Hill equation* is a second-order linear ODE reduced to first order. Consider the following second-order ODE:

|  |  |
| --- | --- |
| (‎1.15) | , |

where is a scalar -periodic function, and are constant parameters, and is the single variable, which, together with its derivative , constructs the state space . The Hill equation is taken from the work on a lunar system of George William Hill (Hill, 1886), (Hill, 1878).

A special case of the Hill equation is called the *Mathieu equation* and is defined by using . Typically, the frequency or, equivalently, the time period s. The Mathieu equation was originally derived in the context of vibrations of stretched elliptical membranes (Mathieu, 1868). Another special case of the Hill equation is called the *Meissner equation* and is defined by . This equation was derived by Meissner while studying the motion of the side rods of a locomotive (Meissner, 1918). The function generates a -periodic square function that changes its sign from +1 over the first half of a period to −1 over the second half. The general *Hill equation* (and *Mathieu equation*) is very difficult to solve analytically and usually requires summing infinite terms, but the *Meissner equation* is easy to solve piecewise by applying a technique for solving LTI systems. □

The purpose of the present study is to find a family of LPTV systems with a finite number of harmonics, and that is analytically solvable by applying a nontrivial technique (i.e., not a technique for solving LTI systems), such as exponentiating the antiderivative of a LPTV matrix.

#### **Example ‎1.4: (Aggarwal & Infante, 1968) and (Rosenbrook, 1963)**

Consider the following system of ODEs, which is a generalization of (Aggarwal & Infante, 1968):

|  |  |
| --- | --- |
| (‎1.16) |  |

This LPTV system is periodic. In this example, we analyze a solution for the transition matrix of the LPTV system as a function of the parameters and . In addition, without loss of generality, we set the initial time [this holds because we can use , , and plug in ] For (Aggarwal & Infante, 1968), the following is obtained:

|  |  |
| --- | --- |
| (‎1.17) |  |

We see that for all , this solution is asymptotically stable. Note that the eigenvalues of are and are time invariant. In addition, for , the eigen values are on the left-hand side (LHS) of the complex plane, which means that they are stable if the eigenvalues are represented as poles of a LTI system. For example, if (Markus & Yamabe, 1960), then the eigenvalues are on the LHS of the complex plane; however, the LPTV system remains unstable. For and , we obtain

|  |  |
| --- | --- |
| (‎1.18) | = |

In this setup, the system is marginally stable because no component in decays or grows over time, but consists of sums of products of two periodic functions: one with frequency that we set, and the other with generated from the solution.

Consider now a generalized example of (Rosenbrook, 1963):[[1]](#footnote-2)

|  |  |
| --- | --- |
| (‎1.19) |  |

The eigenvalues of are , which are stable. However, for , as defined by (Rosenbrook, 1963), we obtain:

|  |  |
| --- | --- |
| (‎1.20) |  |

which has an unstable exponential term . Changing the frequency affects the stability of the solution. For example, produces:

|  |  |
| --- | --- |
| (‎1.21) |  |

which is stable.

To summarize, these examples of LPTV systems have solutions (i.e., the transition matrix) that, on the one hand, are not trivially obtained by using a LTI approach, and LTI stability criteria do not hold for these LPTV systems. In addition, the stability of the solution depends on the frequency . On the other hand, for both the LPTV system and its transition matrix, the number of harmonics remains finite, making them potentially easy to solve.

□

The *state space* of a LPTV system is defined by the set of -periodic matrices , as per Eq. (‎1.12). As stated above, solving a general LPTV system requires first finding from Eq. (‎1.4). Once is obtained, we can obtain the state vector and output vector from Eq. (‎1.13) based on the given initial condition and the control input .

## Literature Survey: Previous Research and Applications

### Stability Analysis of LPTV System

Whether a system is linear or nonlinear, its stability is one of the most significant properties to be analyzed. A necessary condition for quantitative exponential stability of a LTV system is outlined, e.g., in (Lewkowicz, 1999). An unforced LTI system is stable (asymptotically) if and only if all the eigenvalues of the system matrix have a negative real part. However, this statement is false for a more general LTV system and, in particular, for a LPTV system. Two counterexamples are given by (Markus & Yamabe, 1960, pp. 130-131) and (Rosenbrook, 1963, p. 73) (see footnote2 above). In both examples, the LPTV system is represented by a 2×2 periodic matrix with frequency and Fourier-coefficient matrices that are numerically known and have constant eigenvalues with negative real parts, and the solution includes terms that grow exponentially instead of decaying (i.e., are unstable).

These counterexamples (or variations thereof) are discussed in other textbooks [see, e.g., (Amato, 2006) Example 2.2; (Bittanti & Colaneri, 2009) Example 1.1; (Chicone, 2006) Eq. (2.28); (Colonius & Kliemann, 2014) Chapter 6, pages 109–111; (Kelley & Peterson, 2010) Exercise 2.69 and similarly Exercise 2.66; (Khalil, 2002) Example 4.22 and Exercise 10.10(2); (Mathis, 1987) Example 5.13; (Rugh, 1996) Example 8.1]. These counterexamples are also used in some articles to demonstrate stability analyses and solutions of LTV systems and, in particular, of LPTV systems. Aggarwal and Infante (Aggarwal & Infante, 1968) parametrized the example from (Markus & Yamabe, 1960) by scaling the periodic part of the LPTV system. The stability condition is based on this parametrization factor , which is obtained directly from the solution of the LPTV system rather than from the eigenvalues of the LPTV system matrix.

Numerous authors (Colaneri, 2005), (DaCunha, 2004), (Mullhaupt, Buccieri, & Bonvin, 2007), (Varbel, 2020), (Wu, 1974), (Yao, Liu, Sun, Balakrishnan, & Guo, 2012), analyze the stability of LPTV systems and refer to the examples in (Aggarwal & Infante, 1968), (Markus & Yamabe, 1960), or (Rosenbrook, 1963) without suggesting any solution. For example, by using a *logarithmic norm*, (Varbel, 2020) derived a new criterion for uniform exponential stability of a LPTV system without finding the constant matrix or its eigenvalues (the exponents are called the *Floquet characteristic exponents*)*.* Recall that uniform exponential stability is defined as follows:

A LTV system defined by the matrix is:

1. *uniformly stable* iff such that

[or, equivalently, ];

1. *uniformly asymptotically stable* iff such that [or, equivalently, ],

where is some matrix norm operator (e.g., , , ) defined by , where the right-hand side (RHS) refers to a maximal associated vector norm. Note that we can use for a real parameter (Lewkowicz, 1999)*.*

The *logarithmic norm* of a matrix is defined by

|  |  |
| --- | --- |
| (‎1.22) |  |

Example of *logarithmic norms*: ,, and . Note that a *logarithmic norm* may be negative. The main result of (Varbel, 2020) is an estimate of the norm of the transition matrix of a LPTV, , as falling between two integrals of exponentials:

|  |  |
| --- | --- |
| (‎1.23) | and |

where

|  |  |
| --- | --- |
| (‎1.24) | , |

which gives the following stability properties for a LPTV system represented by a -periodic :

* ⇒ LPTV system is *uniformly asymptotically stable*;
* ⇒ LPTV system is *uniformly stable*;
* ⇒ LPTV system is unstable.

However, (Varbel, 2020) refers to a stability analysis without solving the ODE of the LPTV system itself.

### Dynamic Eigen Decomposition of LTV System

van der Kloet and Neerhoff (van der Kloet & Neerhoff, 2004a) (van der Kloet & Neerhoff, 2004b) suggest a generalization of dynamic eigenvalues and a procedure to diagonalize a general LTV system based on the Riccati equation, which is not a linear ODE and might be difficult to solve. This approach is demonstrated on a LPTV system (Aggarwal & Infante, 1968), (Rosenbrook, 1963). The article discusses the use of dynamic eigenvalues along with the Riccati equation and similar approaches (Wang, 2017), and the LPTV system addressed is the same as in (Aggarwal & Infante, 1968). However, instead of using the approach based on the Riccati equation, the LPTV system is solved by using an *auxiliary equation*, which is outlined in ‎APPENDIX D. This requires that an appropriate matrix be selected to find dynamic eigenvalues of the matrix . In addition, is required to be a LTV system with solutions consisting of the corresponding dynamic eigenvectors, which may make it difficult to guess and verify whether it is difficult to implement (see ‎APPENDIX D for more details).

### Small-Perturbation Approach

Yakubovich and Starzhinskii, Chapter 4 (Yakubovich & Starzhinskii, 1975), use a small-parameter approach so that the LPTV system matrix and its solution matrices and can each be decomposed into a power series (infinite or finite) in some small parameter such that:

|  |  |
| --- | --- |
| (‎1.25) |  |

The matrix of coefficients of in and is not a function of ote that for , we have a LTI system [see, e.g., (Prokopenya, 2007) and (Sinha, Pandiyan, & Bibb, 1996)].

### Factorization of Alternative Transition Matrix of LPTV System

Jikuya and Hodaka (Jikuya & Hodaka, 2009), (Jikuya & Hodaka, 2010) suggest factorizing the solution of a LPTV into a periodic function and two matrix exponential functions such that:

|  |  |
| --- | --- |
| (‎1.26) |  |

where commute multiplicatively. is chosen to characterize the stability of the system (such as the matrix given above), and is chosen so that is periodic. This suggestion assumes that the solution is known, although it is not easy to find.

### Generalization of LTI-System Tools for LPTV Systems

Wereley suggests (Wereley, 1991) applying generalized tools to LPTV systems to analyze and control LTI systems (e.g., transfer functions, zeros and poles in and domains). Some examples from (Wereley, 1991) are outlined in ‎APPENDIX E.

## Suggested Approach and Scope of Work

This work focuses on cases in which the coefficients of the Fourier series of [[2]](#footnote-3) are finite polynomials in [[3]](#footnote-4) is a finite polynomial in , and the coefficients of the Fourier series of do not depend on , such that:

|  |  |
| --- | --- |
| (‎1.27) |  |

|  |  |
| --- | --- |
| (‎1.28) |  |

|  |  |
| --- | --- |
| (‎1.29) |  |

|  |  |
| --- | --- |
| (‎1.30) |  |

where and are the coefficients of in the Fourier series (and likewise for ). This setup for , , and serves to compare two coefficients to find and : Fourier coefficients and powers of . The Fourier coefficients of must not depend on to assure that the terms and are polynomials in (i.e., without negative powers), and also to assure a continuity in the parameter in , , and [sometimes this condition is denoted , , and to emphasize the dependency on ].

# Floquet Theory and its Properties

## Overview

This chapter introduces *Floquet theory*, which is the main theorem for factorizing the transition matrix of a LPTV system. In addition, we outline some properties derived from *Floquet theory*. This chapter is organized as follows: Section ‎2.2 presents *Floquet theory* itself. Section ‎2.3 shows how to transform a LPTV system into a LTI system when the transitions matrix of the LPTV system is known (or at least the periodic part is known). Section ‎2.4 shows some additional properties of a LPTV system matrix when its trace average is zero and how zero trace can be obtained for any LPTV system. Section ‎2.5 demonstrates that decomposition of a LPTV system’s transition matrix into a periodic part and a constant part is not unique. Section ‎2.6 combines Sections ‎2.2 to ‎2.5 to present an alternative version of *Floquet theory*. Finally, the chapter is summarized in Section ‎2.7.

## Floquet Theory

**Theorem ‎2.1:** Suppose is a solution to the ODE of the following LPTV system defined by the -periodic matrix :

|  |  |
| --- | --- |
| (‎2.1) |  |

Then, according to *Floquet theory*, the transition matrix of can be decomposed into

|  |  |
| --- | --- |
| (‎2.2) |  |

where is a constant matrix for all and is a non-singular -periodic matrix [i.e., ].□

**Lemma ‎2.2:** The transition matrix of a LPTV system is simultaneously -periodic in both time inputs such that:

|  |  |
| --- | --- |
| (‎2.3) |  |

**Proof:** By using the periodicity of and the fact that is constant in time, we obtain:

.

∎

For **Lemma ‎2.3** below, we denote the average of as follows:

|  |  |
| --- | --- |
| (‎2.4) |  |

**Lemma ‎2.3:** If a LPTV system is given by Eq. (‎2.1) and the transition matrix by Eq. (‎2.2), then:

|  |  |
| --- | --- |
| (‎2.5) |  |

**Proof:** According to (Rugh, 1996), Property 4.9,

By using Eq. (‎2.2) to calculate the RHS of the above, we obtain:

This last equation holds for all , and especially for . In addition, since , then:

By adding to both sides and using the average of [Eq. (‎2.4)], we conclude that:

∎

By plugging Eq. (‎2.2) into Eq. (‎1.4), we obtain the following matrix differential equation:

|  |  |
| --- | --- |
| (‎2.6) |  |

Two equivalent equations for Eq. (‎2.6) are given by solving for and for :

|  |  |
| --- | --- |
| (‎2.7) |  |

and

|  |  |
| --- | --- |
| (‎2.8) |  |

Equation (‎2.7) stipulates that, given the matrices and , we can generate a LPTV system matrix . However, the goal is to solve in the opposite direction; i.e., given a matrix , we need to find the matrices and that construct the transition matrix of given by Eq. (‎2.2). In this work, we use Eq. (‎2.7) to produce a LPTV system matrix for use as a tool to explore the properties and solutions of LPTV systems.

Equation (‎2.8) says that, for a given LPTV system matrix , a periodic matrix inserted into the RHS of Eq. (‎2.8) should produce a constant matrix on the LHS if is correct. Equation (‎2.8) is a useful tool to double-check solutions and to calculate the constant matrix when information is obtained regarding the structure of .

## Relation between LPTV and LTI Systems

The *Lyapunov reducibility theorem* [see Chapter 2, Section 2.4 of (Yakubovich & Starzhinskii, 1975)] gives:

|  |  |
| --- | --- |
| (‎2.9) |  |

Plugging this equation into Eq. (‎2.1) and using Eq. (‎2.8) produces a LTI system,

|  |  |
| --- | --- |
| (‎2.10) |  |

that connects the LPTV system represented by the system matrix to its LTI system represented by the system matrix . Unfortunately, neither nor is known. If we try an incorrect periodic part , Eq. (‎2.10) becomes a new LPTV system with a new periodic system matrix defined by the RHS of Eq. (‎2.8). Furthermore, we would like to find a case where we can obtain information from regarding the structure and values of and by mathematical manipulation.

## Zero Average LPTV System Matrix

This section explores properties stemming from the trace of the LPTV system matrix that translate into properties of and thereby help us to find . Taking the trace of Eq. (‎2.7), subtracting from it Eq. (‎2.5), and using the trace property , we obtain the following identity:

|  |  |
| --- | --- |
| (‎2.11) |  |

From Eq. (‎2.5) we conclude that

|  |  |
| --- | --- |
| (‎2.12) |  |

Therefore, we conclude that

|  |  |
| --- | --- |
| (‎2.13) |  |

If the is identically zero, then the integral over the trace within each interval is zero, so exponentiating this integral gives unity, which implies that for all . Since are arbitrary, we must have for a constant that does not depend on either or . If is a function of that is not identically zero, then the term is a non-constant function of , so is a non-constant function of

The above property motivates us to find a family of LPTV system matrices where the periodic part of the transition matrix has a constant determinant, which should give us significant information for computing . We will see that the value of this determinant is arbitrary because, among other things, can be defined up to a scaling factor. In addition, a mathematical manipulation can always produce a constant determinant for .

## Nonunique Decomposition of LPTV Transition Matrix

On the one hand, a unique solution to a LTV ODE that depends on the transition matrix exists for every initial condition. On the other hand, the decomposition of the transition matrix into the product of the periodic-function matrix and the constant matrix is not unique. This is shown in the following sections.

### Similarity Transformation of *R*

Suppose that a constant matrix is decomposed into another similar constant matrix so that:

|  |  |
| --- | --- |
| (‎2.14) |  |

where is some arbitrary non-singular constant matrix, then the transition matrix can be calculated from Eq. (‎2.2) as follows:

|  |  |
| --- | --- |
| (‎2.15) | . |

Therefore, the periodic function matrix and the constant matrix can also be used to construct the transition matrix for any non-singular constant matrix (i.e., the decomposition of the transition matrix is not unique).

### Similarity Transformation of *A*(*t*)

We may consider changing the state variable. Suppose is a solution to a LPTV system [see Eq. (‎2.1)] and consider the following change of state variable:

|  |  |
| --- | --- |
| (‎2.16) |  |

where is an arbitrary constant, non-singular matrix. We can then produce a new LPTV system in so that:

|  |  |
| --- | --- |
| (‎2.17) |  |

By multiplying Eq. (‎2.6) subjected to the transition matrix by , we obtain:

|  |  |
| --- | --- |
| (‎2.18) |  |

We thus conclude that the periodic matrix and the constant matrix constitute the solution for decomposing the transition matrix of the new system matrix . Therefore, the solution of is:

|  |  |
| --- | --- |
| (‎2.19) |  |

Equation (‎2.17) gives the solution in terms of :

|  |  |
| --- | --- |
| (‎2.20) | ⟹. |

To summarize, we have a fair degree of freedom to achieve the matrices and , for example, by assuming that is of canonical form, or or equals any nonsingular matrix that can be defined up to, e.g., a scaling factor or a rotation. In addition, we may consider transforming the system matrix to simplify the calculation of the required transition matrix.

In the next section, we show how to obtain a transition matrix by using the principles of this section, the previous section ‎2.4 (using a zero-trace system matrix), and Eq. (‎1.5).

## Floquet Theory: Zero-Trace System Matrix

**Theorem ‎2.4:** If is a LPTV system matrix in the sense of Eq. (‎2.1), then the transition matrix can be decomposed as follows:

|  |  |
| --- | --- |
| (‎2.21) |  |

where is a periodic matrix with , and is a constant matrix with .

**Proof:** We use the notion of shifting the LTV system matrix diagonal [see, e.g., (Lewkowicz, 1999) and Eq. (‎1.5)] and denote , . We know from *Floquet theory* that, in general, a transition matrix of has the form . Given that , the following holds:

1. from Eq. (‎2.4): ;
2. from Eq. (‎2.5) and item 1:;
3. from Eq. (‎2.13) and item 1,

.;

1. from Eq (‎1.5),

.

Finally, Eq. (‎2.21) is obtained given the conditions and .∎

Consider decomposing as follows:

|  |  |
| --- | --- |
| (‎2.22) |  |

where is the constant part of [i.e., the average of ] and is the periodic part of with . The anti-derivative of is denoted [[4]](#footnote-5)

|  |  |
| --- | --- |
| (‎2.23) |  |

We can write Eq. (‎2.23) as follows:

|  |  |
| --- | --- |
| (‎2.24) |  |

so that:

|  |  |
| --- | --- |
| (‎2.25) | and |

are the matrices to construct the transition matrix

## Summary

This chapter summarizes *Floquet theory*. The main theorem factorizes the transition matrix

The next chapter presents some examples that contain a finite or infinite number of harmonics, and the frequency as a free parameter in LPTV systems is used to examine how affects the solution for a LPTV system (e.g., stability).

# Examples of Analyses of LPTV Systems

## Overview

A variety of examples of matrices of LPTV systems and their corresponding matrices and obtained using *Floquet theory* are now shown. In all examples, the frequency of the LPTV system matrix is set to be some arbitrary constant . Without loss of generality, it is assumed that (otherwise the parity of sine and cosine could be used to represent a LPTV system with a positive frequency). Our examples are also limited to those whose periodic matrices and are constructed by applying elementary functions,[[5]](#footnote-6) such as trigonometric polynomials formed by .

This chapter is organized as follows: Section ‎3.2 presents several examples of LPTV system matrices and their corresponding matrices and categorized into four cases according to the number of harmonics (infinite or finite) in and . The motivation for this approach is to investigate situations in which both and have a finite number of harmonics. Section ‎3.3 outlines the notion of a frequency as a free parameter in LPTV systems. The examples below focus on the simple case in which the Fourier coefficients of and are linear functions of , whereas the Fourier coefficients of are constant. Finally, the chapter is summarized in Section ‎3.4.

## Number of Harmonics in LPTV Systems and in the Periodic Parts of their Transition Matrices

### General

Let *L* be the number of harmonics of the matrix for a LPTV system and *p* be the number of harmonics of the periodic part of its transition matrix. The following cases are considered:

* **Case 1**: is infinite and is infinite;
* **Case 2**: is infinite and is finite;
* **Case 3**: is finite and is infinite;
* **Case 4:**  is finite and is finite.

We search for a family of matrices of LPTV systems that correspond to Case 4 (finite ) and for a procedure that uses *Floquet theory* to calculate the periodic matrix (with finite ) and constant matrix to construct the transition matrix of this family. Each case itself may have some subcases, such as generating an infinite trigonometric polynomial by some elementary operation (e.g., dividing a finite trigonometric polynomial by another finite trigonometric polynomial, or exponentiating a finite trigonometric polynomial), or by some non-elementary operation (e.g., performing a periodic continuation of a non-periodic function). From *Floquet theory* [Eq. (‎2.7)], the condition . [Eq. (‎2.13)], and ‎APPENDIX A (specifically **Lemma ‎A.1**, **Lemma ‎A.2**, and **Lemma ‎A.3**), we assert that, if has a finite number *p* of harmonics, then has at most harmonics because the sum has harmonics, has at most harmonics, and their product can produce at most harmonics. In particular, if , then has at most harmonics.

Consider solving a LPTV system by using Eq. (‎2.6). Since the RHS of this equation [i.e., ] has harmonics, then the LHS [i.e., ] must have exactly harmonics. Otherwise the Fourier coefficients would not match for all . Normally, the number of harmonics of is given, whereas that of is unknown. We search for a family of LPTV system matrices with a finite number *L* of harmonics that satisfies the conditions above [i.e., is periodic with a finite number *p* of harmonics to be determined]. We hypothesize that .

### Examples

We explore here some examples of cases 1–4.

**Table ‎3‑1** Examples of 2×2 LPTV Systems. Case 1: L is infinite, p is infinite.

|  |  |  |  |
| --- | --- | --- | --- |
| Row |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

**Table ‎3‑2** Examples of 2×2 LPTV Systems. Case 2: L is infinite, p is finite.

|  |  |  |  |
| --- | --- | --- | --- |
| Row |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

**Table ‎3‑3** Examples of 2×2 LPTV Systems. Case 3: L is finite, p is infinite.

|  |  |  |  |
| --- | --- | --- | --- |
| Row |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |

**Table ‎3‑4** Examples of 2×2 LPTV Systems. Case 4: L is finite, p is finite.

|  |  |  |  |
| --- | --- | --- | --- |
| Row |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  |  |  |  |
|  | + |  |  |
|  | + |  |  |
|  | Generalized cases: |  |  |
|  | + |  |  |

In the above examples, let , *a*, *b*, *c*, and *d* be real parameters that define the matrices *A*(*t*) of LPTV systems and their corresponding matrices . Note that the examples in rows A and B of **Tables ‎3** and **4** are equivalent in the sense that matrices of the LPTV systems are the same. This equality holds because in both rows use the notion discussed in Section ‎2.5.1; namely, and with . We search for a family of real LPTV systems similar to those of **Tables ‎3** and **4**, so that this family will be applicable for every real frequency .

## Notions of Frequency ω as a Free Parameter in LPTV Systems

### General

We suggest using the frequency *ω* of a LPTV system as a free parameter (instead of plugging in a fixed value). This approach will allow for an examination of a family of solutions for LPTV systems and explore the system properties based on a variety of values of . Such properties include stability criteria, low-frequency behavior (approaching a LTI system), and high-frequency behavior.

We limit ourselves to the case in which each Fourier coefficient of is an ordinary polynomial in , so that is a polynomial in , but the Fourier coefficients of [and, thus, also of ] are not polynomials in . This is done to simplify the problem and to assure the continuity for all of , , , and . For this demonstration, the polynomial in in the examples is linear affine in .

### Examples

#### **Example ‎3.1: Rows ‎J of Tables ‎3 and 4**

Consider the following real LPTV system matrix:

|  |  |
| --- | --- |
| (‎3.1) |  |

The transition matrix of this LPTV system is constructed by the following pair of matrices and :

|  |  |
| --- | --- |
| (‎3.2) |  |

|  |  |
| --- | --- |
| (‎3.3) |  |

From Eq. (‎3.3), it follows that stability imposes the following two conditions:

1. ;
2. .

Note that , so the transition matrix is

|  |  |
| --- | --- |
| (‎3.4) |  |

This LPTV system is stable if and only if the real part of eigenvalues of are negative, which depends on the frequency (and likewise for ). Expressing in a closed form in and solving a LPTV system for some general frequency is equivalent to the finding a diagonal version of (which might be complex) for a specified frequency. Consider the same LPTV system, but at low frequency (). On the one hand,

|  |  |
| --- | --- |
| (‎3.5) |  |

which is a LTI system with transition matrix:

|  |  |
| --- | --- |
| (‎3.6) |  |

On the other hand, if we evaluate the transition matrix of the original LPTV system at low frequency (i.e., approaches zero), we obtain:

|  |  |
| --- | --- |
| (‎3.7) |  |

This example of a LPTV system matrix is a generalization of the examples of (Aggarwal & Infante, 1968), (Markus & Yamabe, 1960), and (Rosenbrook, 1963), which are cited in a number of articles and books, as previously stated in Section ‎1.4. In addition, this LPTV system matrix can be generated according to Eq. (6.2.6) of (Colonius & Kliemann, 2014) with the following setup:

|  |  |
| --- | --- |
| (‎3.8) |  |

and can now be written in terms of and as follows:

|  |  |
| --- | --- |
| (‎3.9) |  |

□.

#### **Example ‎3.2: (Aggarwal & Infante, 1968), (Markus & Yamabe, 1960), and rows B of Tables ‎3 and 4**

Consider the following real LPTV system matrix (cf. rows B of **Tables ‎3 and 4**):

|  |  |
| --- | --- |
| (‎3.10) |  |

with the following matrices and as its solution:

|  |  |
| --- | --- |
| (‎3.11) |  |

|  |  |
| --- | --- |
| (‎3.12) |  |

From Eq. (‎3.12), it follows that stability imposes the following two conditions:

1. ;
2. .

Note that this example is a particular case of **Example ‎3.1** (rows J of **Tables ‎3 and 4**) with some parameters adjusted. Moreover, this example is a general case of (Aggarwal & Infante, 1968) with and (Markus & Yamabe, 1960) with and ). The eigenvalues of depend on and and are given by

|  |  |
| --- | --- |
| (‎3.13) |  |

Previous works discuss the stability of this LPTV system based on a fixed value of (=1) with *a* varying. For , we have a diagonal matrix:

,

which is a trivial case of matrix exponentiation, i.e.,

.

Instead of varying and fixing , we examine the opposite situation by fixing and analyzing the stability of the LPTV system upon varying In this case, the eigenvalues of , which depend on, are:

|  |  |
| --- | --- |
| (‎3.14) |  |

Where:

|  |  |
| --- | --- |
| (‎3.15) | . |

Figure 3-1 and Table 3-5 summarize the characteristics of and as functions of for frequencies determined by:

* the roots of (), which gives conditions on for complex eigenvalues (i.e., );
* the roots of [], which gives conditions on for a stable (i.e., for ).



**Figure ‎3‑1** Eigenvalue of as functions of frequency (Markus & Yamabe, 1960). (a) λ1; (b) λ2.

**Table ‎3‑5** Stability status of etR for different values of (Markus & Yamabe, 1960).

|  |  |  |  |
| --- | --- | --- | --- |
|  |  | Stability Status | Frequency Set |
| 0 |  | Stable |  |
| 0.25 |  | Stable |  |
| 0.28 |  | Stable |  |
|  |  | Marginally  Stable |  |
| 1 |  | Unstable |  |
|  |  | Marginally  Stable |  |
| 1.72 |  | Stable |  |
| 1.75 |  | Stable |  |
| 2 |  | Stable |  |

The eigenvalue frequencies are symmetric about some frequencies ( in this example), so that is reflected about (i.e., ). The point is thus a critical point in the sense that the real part of is maximal (=0.5) and the real part of is minimal (=−1). Given the complex conjugate pair property, the imaginary part varies when the real part is constant (−0.25 in this example), and the imaginary part is zero when the real part varies. Note that, referring to rows C of **Tables ‎3** and **4**, we have the following real LPTV system matrix:

|  |  |
| --- | --- |
| (‎3.16) |  |

with the following matrices and as its solution:

|  |  |
| --- | --- |
| (‎3.17) |  |

|  |  |
| --- | --- |
| (‎3.18) |  |

This result comes from taking the matrices and from rows B of **Tables ‎3** and **4**, evaluating at ], and applying to and to obtain [for ,]. This demonstrates that may be independent of frequency (i.e., constant); however, this is not true in general. Moreover, assume for some fixed frequency that the matrices and form a solution for a LPTV system with the same basic fixed frequency . If changes to a new value from the set of frequencies defined by the Fourier coefficients of , then changing to in the Fourier coefficients of without changing provides a solution to the LPTV system at the new frequency. This is done by applying Eq. (‎2.7) to obtain a potential solution for the LPTV system at the new frequency, which may produce terms with frequencies that are multiples of .

□

## Summary

This chapter outlines examples of

The next chapter presents a Fourier analysis of a LPTV system (cosine-sine form) and derives an algebraic equation with the Fourier coefficients of the matrices and as unknown variables.

# Fourier Analysis of LPTV Systems

## Overview

This chapter implements a Fourier analysis of LPTV systems [presented by the *T*-periodic matrix ] and their -periodic part of the solution provided by *Floquet theory*. We develop the equation under the assumption that the LPTV system is real [i.e., ], so it is convenient to use the cosine-sine Fourier decomposition to find a pair of real matrices, , as a solution per *Floquet theory*. The cosine-sine Fourier decomposition gives matrices with real Fourier coefficients separated according to their parity (even or odd).

For comparison, ‎APPENDIX B outlines a case that applies a complex Fourier decomposition to a LTPV system and its *Floquet theory* solution. ‎APPENDIX C, Section ‎C.1 outlines how a complex LPTV system can be converted to a real LPTV system based on the isomorphism , and Section ‎C.2 outlines how to decompose the state vector into even and odd parts and then rewrites the LPTV system matrix with even and odd matrix blocks.

## Cosine-Sine Fourier Analysis of LPTV Systems

### General

Suppose that the periodic matrices and are solutions to Eq. (‎2.6) and are used to construct the transition matrix . As per ,, we use the following cosine-sine Fourier decomposition for and :

|  |  |
| --- | --- |
| (‎4.1) |  |

|  |  |
| --- | --- |
| (‎4.2) |  |

where (), and () are real Fourier coefficients (in general, they may be complex) of matrices of [] in and , respectively.

We need to decompose all terms in Eq. (‎2.6) into even functions [i.e., coefficients of and 1] and odd functions [i.e., coefficients of ], and likewise also decompose and themselves. The product includes a linear combination of products of cosine and sine factors, so the following trigonometric identities will be useful to convert these products to sums:

|  |  |
| --- | --- |
| (‎4.3) |  |
| (‎4.4) |  |
| (‎4.5) |  |

### Even-Odd Decomposition

Any function can be decomposed into a sum of an even function and an odd function :

|  |  |
| --- | --- |
| (‎4.6) |  |

where

|  |  |
| --- | --- |
| (‎4.7) |  |

and

|  |  |
| --- | --- |
| (‎4.8) |  |

By using the above even-odd decomposition property, is decomposed as follows:

|  |  |
| --- | --- |
| (‎4.9) |  |

where

|  |  |
| --- | --- |
| (‎4.10) |  |

and likewise for . Plugging the above even-odd decomposition of Eq. (‎4.9) [for both and ] into Eq. (‎2.6) gives:

|  |  |
| --- | --- |
| (‎4.11) |  |

Note that the derivative of an even function is an odd function and vice versa. In addition, multiplication of even or odd functions gives even functions, and multiplication of even by odd functions (or vice versa) gives odd functions. Therefore, we obtain:

|  |  |
| --- | --- |
| (‎4.12) |  |

as even terms, and:

|  |  |
| --- | --- |
| (‎4.13) |  |

as odd terms.

The terms of the form and on the LHS of Eqs. (‎4.12) and (‎4.13) are linear combinations of products of a pair of cosine and/or sin terms, so we use the trigonometric identities (‎4.3)–(‎4.5) to convert these products to sums. Re-indexing the result and using the even-odd properties from Eq. (‎A.5) gives:

|  |  |
| --- | --- |
| (‎4.14) |  |

|  |  |
| --- | --- |
| (‎4.15) |  |

The other terms on the RHS of Eqs. (‎4.12) [] are already linear combinations of , and those on the RHS of Eq. (‎4.13) are linear combinations of . By comparing the coefficients of and , we obtain:

|  |  |
| --- | --- |
| (‎4.16) |  |

|  |  |
| --- | --- |
| (‎4.17) |  |

To eliminate negative indices of and , we re-index, using the even-odd properties from Eq. (‎A.5). Upon collecting the coefficients of and and multiplying by two, we obtain, for ,

|  |  |
| --- | --- |
| (‎4.18) |  |

for the even (cosine) terms, and:

|  |  |
| --- | --- |
| (‎4.19) |  |

for the odd (sine) terms. For the even-odd properties [see Eq. (‎A.5)] lead to:

|  |  |
| --- | --- |
| (‎4.20) |  |

fthe . The sine (odd) terms give zero:

|  |  |
| --- | --- |
| (‎4.21) |  |

Equations (‎4.18)–(‎4.20) can be represented as a semi-infinite block-system of equations:

|  |  |
| --- | --- |
| (‎4.22) |  |

where:

|  |  |
| --- | --- |
| (‎4.23) |  |

|  |  |
| --- | --- |
| (‎4.24) | = |

In Eq. (4.24),

|  |  |
| --- | --- |
| (‎4.25) |  |

is the block matrix in row , column that multiplies the element .

## Summary

This chapter presents a Fourier analysis of a LPTV system in cosine-sine form and derives algebraic equations containing the matrix and the Fourier coefficients of as unknown variables. Referring to ‎APPENDIX B, we note a similarity between the structures of the algebraic equations in the real-imaginary decomposition and in the even-odd decomposition.

In ‎CHAPTER 3, we addressed the case where both and have a finite number of harmonics and observed that the Fourier coefficients of are linear functions of , those of are constant (independent of ), and those of are linear functions of . This fact motivates the next chapter, in which we compare powers of . This means that we treat the frequency as a free parameter in LPTV systems, and not as a fixed number.

# LPTV Systems with Coefficients as Polynomials in ω

## Overview

This chapter introduces a family of periodic matrices for with a finite number of harmonics, so that each Fourier coefficient of is a polynomial in . As per *Floquet theory*, we search for some conditions so that the corresponding matrices produce a transition matrix such that the number of harmonics of is also finite.

**Remark**: To simplify the problem, we assume, without loss of generality, that . If not, then we can apply the trace-zero system matrix and use the trace version of *Floquet theory* **(Theorem ‎2.4**) so that ., , and recast as so that .

This chapter is organized as follows: Section ‎5.2 describes how to express Fourier coefficients as polynomials in for a cosine-sine Fourier series of a LPTV system matrix and the periodic part of its solution. Section ‎5.3 relates a LPTV system with Fourier coefficients expressed as polynomials in and its LTI version for the LPTV system evaluated at . Section ‎5.4 transforms a LPTV system so that, at , we obtain a canonical form of a LTI system (e.g., Jordan block diagonal). Section ‎5.5 compares powers of to solve a LPTV system. Finally, Section 5.6 summarizes the chapter.

## Cosine and Sine Fourier Coefficients as Polynomials in ω

Suppose that a LPTV system matrix is defined as follows:

|  |  |
| --- | --- |
| (‎5.1) |  |

|  |  |
| --- | --- |
| (‎5.2) |  |

where is the size of the matrix , is the degree of the polynomial in , , is the th Fourier series such that and are Fourier coefficients that do not depend on , and is the number of the harmonics of . Inserting Eq. (‎5.2) into Eq. (‎5.1) and changing the order of summation gives:

|  |  |
| --- | --- |
| (‎5.3) |  |

For convenience, we denote the Fourier coefficients of as follows:

|  |  |
| --- | --- |
| (‎5.4) |  |

Inserting in the new definition above for into Eq. (‎2.6) gives:

|  |  |
| --- | --- |
| (‎5.5) |  |

It might be convenient to have a degree- polynomial in powers of , so we assume that is a degree- polynomial in , but not . Thus,

|  |  |
| --- | --- |
| (‎5.6) |  |

As done at the beginning of this chapter, we use *Floquet theory* (**Theorem ‎2.4**) so that and , which gives for . By using the chain rule, we obtain , which provides a linear term (proportional to ) on the LHS of Eq. (‎5.5) so that both sides of Eq. (‎5.5) can be presented as a degree- polynomial in . Combining the above relations with Eqs. (‎4.18) and (‎4.19) gives the new cosine and sine coefficients for comparison. the cosine (even) terms are:

|  |  |
| --- | --- |
| (‎5.7) |  |

and the sine (odd) terms are:

|  |  |
| --- | --- |
| (‎5.8) |  |

Comparing powers of gives the following set of algebraic equations (for and ):

|  |  |
| --- | --- |
| (‎5.9) |  |

for the cosine (even) and terms, and:

|  |  |
| --- | --- |
| (‎5.10) |  |

for the sine (odd) and terms. In Eqs. (5.9) and (5.10), is the Kronecker delta function and multiplies the term . By using Eqs. (‎4.22)−(‎4.25) and comparing powers of , Eqs. (‎5.9) and (‎5.10) can be presented as a matrix-vector form of algebraic equations (i.e., ).

We thus need to solve Eqs. (‎5.9) and (‎5.10) for each term in (for each ) and for each and (for each ). However, assuming that each and is independent of , with if we find a set of solutions that is compatible with some solution for an abitrary .

Moreover, suppose that, for an arbitrary , we find a solution and a concurrent set . Since we know all the terms in and , we can construct the periodic part of the solution, and we can use Eq. (‎2.8) to find the constant part . **Theorem ‎2.4** of *Floquet theory* indicates that our solution is correct:

1. .;
2. the RHS of Eq. (‎2.8) [
3. the result for in item 2 above is the sum [see Eq. (‎5.6)], so
   1. for all ;
   2. the term (that was calculated to find the set ) is the same term calculated from the sum for .

## Relation to LTI Systems

We now consider , in which case the LPTV system is not periodic. In this case, we find that the system becomes a LTI system. We want to find the relationship between the original LPTV system with its transition matrix (for some ) and the related LTI system with its transition matrix (when ). We first need to examine the scalar term in **Theorem ‎2.4** of *Floquet theory* to verify that it does not diverge when for fixed times and .

**Lemma ‎5.1:** For all constants

and .

This can be trivially proved by using elementary trigonometric identities and limits:

This is not detailed in this work

□.

**Corollary ‎5.2**: If has the form:

,

where none of the terms or depend on , then:

Therefore, on the one hand:

but, on the other hand,

We conclude that, for a scalar LPTV system , the transition function is a continuous function of and, in particular, at the point . In addition, produces a scalar LTI system with a transition function . □

We now extend the notion of this corollary to the matrix case of LPTV systems and their solution (see **Theorem ‎2.4**of *Floquet theory*). For a frequency analysis, we change the notation “” used to indicate time dependence to a notation using time and frequency, [for example, ]. Suppose that:

|  |  |
| --- | --- |
| (‎5.11) |  |

and, without loss of generality (see the beginning of the chapter),

|  |  |
| --- | --- |
| (‎5.12) |  |

We denote the *Floquet theory* solution by using the periodic matrix and the constant matrix as follows:

|  |  |
| --- | --- |
| (‎5.13) |  |

Where:

|  |  |
| --- | --- |
| (‎5.14) | , , |

so that the transition matrix is:

|  |  |
| --- | --- |
| (‎5.15) |  |

On the one hand, if we insert into Eqs. (‎5.11) and (‎5.12), we obtain:

|  |  |
| --- | --- |
| (‎5.16) |  |

which is a constant matrix that can represent a LTI system with a transition matrix:

|  |  |
| --- | --- |
| (‎5.17) |  |

On the other hand, if we insert into Eqs. (‎5.13)–(‎5.15), we have the following equation involving and :

|  |  |
| --- | --- |
| (‎5.18) |  |

Where:

|  |  |
| --- | --- |
| (‎5.19) | , , , |

so that the transition matrix is:

|  |  |
| --- | --- |
| (‎5.20) |  |

Note that, in solving LPTV systems, we have some degree of freedom to determine (up to a scaling factor, an invertible rotation or reflection transformation, etc.), so we may determine by using one of the following options:

1. Choose to have some canonical form (e.g., a real Jordan matrix block) so that is constructed such that for an invertible matrix , and then find such that .
2. Set and find such that .

Using either option 1 or option 2, we obtain:

|  |  |
| --- | --- |
| (‎5.21) |  |

which is the transition matrix generated by the LPTV system matrix and then evaluated at [denoted ; see Eq. (‎5.20)]. is the same as the transition matrix generated by the related LTI system matrix [denoted ; see Eq. (‎5.17)].

## Transformation of LPTV Matrix to Canonical Form at ω = 0

Suppose that is a LPTV system matrixthat is continuous in frequency (e.g., with Fourier coefficients that are polynomial in ). Consider the eigen decomposition of :

|  |  |
| --- | --- |
| (‎5.22) |  |

where has a real Jordan canonical block form [i.e., we can represent eigenvalues of ], and is the corresponding eigenvector matrix. By using the invertible matrix , we transform into a new similar matrix:

|  |  |
| --- | --- |
| (‎5.23) |  |

which has the canonical form at [i.e., ]. Given that and are similar matrices, they have the same trace. We denote . If , we can zero the trace by using:

|  |  |
| --- | --- |
| (‎5.24) |  |

Note that we can interchange the eigen decomposition and the trace-zeroing; that is, we can first do the eigen decomposition as per Eq. (‎5.23) and then zero the trace as per Eq. (‎5.24), or vice versa. The eigenvector matrix remains the same, as required, and we have a new canonical form for such that as required. Applying *Floquet theory* to the LPTV system matrix , we have the matrices and such that

|  |  |
| --- | --- |
| (‎5.25) |  |

where is the periodic matrix of the solution with , and is the constant part of the solution with .. can be chosen with the following properties:

|  |  |
| --- | --- |
| (‎5.26) |  |

When using in Eq. (‎5.25), we obtain a LTI system represented by the constant matrix . In addition, is a constant matrix at , so choosing is compatible with the equations above. The motivation to perform these transformations is to simplify the structure of the matrices and and to simplify the procedure to find them. In particular, if the Fourier coefficients of are polynomial in so that is polynomial in , but the Fourier coefficients of are independent of , it would be convenient to find the Fourier coefficients of by comparing powers of between Eqs. (‎5.9) and (‎5.10) and then solving for by using Eq. (‎5.25).

From the properties of Eqs. (‎2.20) and (‎2.21), the transition matrix can be decomposed as follows based on the solution for and :

|  |  |
| --- | --- |
| (‎5.27) |  |

The factor is responsible for zeroing the trace for Eq. (‎5.24), and inverts the canonical form decomposition [see Eq. (‎5.23)]. Note that:

|  |  |
| --- | --- |
| (‎5.28) | and |

are the matrices used to obtain the transition matrix . Based on Eqs. (‎2.23) and (‎2.24), we can decompose the scalar function into a periodic part [denoted by with anti-derivative ; see footnote5] and a constant part (denoted by ). We then construct the matrices

|  |  |
| --- | --- |
| (‎5.29) | and |

For the transition matrix , see **Theorem ‎2.1**of *Floquet theory*.

The next section shows several examples of solving a LPTV system by comparing powers of .

## Examples

#### **Example ‎5.3: Rows J of Tables ‎3 and4**

Consider the following real LPTV system matrix:

|  |  |
| --- | --- |
| (‎5.30) |  |

In general, . However, based on Section ‎2.6, we can solve for the case where and apply the following equation to obtain:

|  |  |
| --- | --- |
| (‎5.31) |  |

The factor plays the role of the function (with ). We now solve for and omit the hat sign,

|  |  |
| --- | --- |
| (‎5.32) |  |

We assume that the transition matrix is decomposed according to *Floquet theory* into a periodic part and a constant part such that and for all (i.e., ). The Fourier coefficients of , and , are defined for but not for , so we assume that the Fourier coefficients of , and , are defined for . Recall that the superscript denotes the power of (i.e., and are the Fourier coefficients of for ). By evaluating at we obtain:

|  |  |
| --- | --- |
| (‎5.33) |  |

We can use this information to obtain because [Eq. (‎5.18)]. Note that the eigenvalues of are:

|  |  |
| --- | --- |
| (‎5.34) |  |

which might be complex. Conversely, is a square matrix,

|  |  |
| --- | --- |
| (‎5.35) |  |

which has two identical eigenvalues . Therefore, we use and . By using Eqs. (‎4.22)–(‎4.25), we have the following algebraic matrix-vector form:

|  |  |
| --- | --- |
| (‎5.36) |  |

With:

|  |  |
| --- | --- |
| (‎5.37) |  |

|  |  |
| --- | --- |
| (‎5.38) |  |

where and the terms and are unknown and must be found. To simplify the structure, we solve for the coefficient of in Eqs. (‎5.36)–(‎5.38) (or, equivalently, consider and retain all coefficients of ; e.g., ), and obtain:

|  |  |
| --- | --- |
| (‎5.39) |  |

By multiplying Eq. (5.39) by  from the left and using the associative property of matrix multiplication, we have a new LHS multiplied by from the left and a new RHS multiplied by , such that:

|  |  |
| --- | --- |
| (‎5.40) |  |

where , and is:

|  |  |
| --- | --- |
| (‎5.41) |  |

The eigen decomposition of is given by:

|  |  |
| --- | --- |
| (‎5.42) |  |

where:

|  |  |
| --- | --- |
| (‎5.43) |  |

Given that , we construct by a linear combination of the columns of , which are eigenvectors corresponding to the eigenvalue . If we set

|  |  |
| --- | --- |
| (‎5.44) |  |

with

|  |  |
| --- | --- |
| (‎5.45) |  |

then we use the following guess for :

|  |  |
| --- | --- |
| (‎5.46) |  |

Note that the above guess for satisfies and , which is consistent with our assumptions.

We now reintroduce the hat superscript [e.g., ]. Inserting our guess for  into the RHS of Eq. (‎2.8), we can check if the value of the constant part of the solution is valid:

|  |  |
| --- | --- |
| (‎5.47) |  |

We find that is not a function . In addition, the coefficient of in the expression for is identical to our assumption . Therefore, the matrices and are a solution for constructing the transition matrix of the shifted version of [i.e., , see Eq. (‎5.32)]. To obtain the solution of the original LPTV system , we use Eq. (‎2.25); given , the following matrices and are a solution to construct :

|  |  |
| --- | --- |
| (‎5.48) |  |

which is consistent with the data given in rows J of **Tables ‎3** and **4**. ∎

The procedure used here to solve this parametric example can be used to solve a family of LPTV systems from the literature (Aggarwal & Infante, 1968), (Markus & Yamabe, 1960), (Rosenbrook, 1963). Note that a numerical example does not require the eigen decomposition in Eq. (‎5.40); the eigen decomposition in Eq. (‎5.39) suffices. In the following examples, we solve LPTV systems with a larger number of harmonics in and and finish with a 3×3 example.

#### **Example ‎5.4: Rows E of Tables ‎3 and 4**

Consider the following real 2×2 matrix for a LPTV system :

|  |  |
| --- | --- |
| (‎5.49) | , |

which has harmonics. By writing , we obtain:

|  |  |
| --- | --- |
| (‎5.50) |  |

Note that and ; thus, the trace need not be zeroed. In addition, the matrix evaluated at ,

|  |  |
| --- | --- |
| (‎5.51) |  |

is decomposed as follows:

|  |  |
| --- | --- |
| (‎5.52) |  |

where

|  |  |
| --- | --- |
| (‎5.53) |  |

We assume that the transition matrix is decomposed according to *Floquet theory* into a periodic part and a constant part Since has harmonics and is of size , we hypothesize that has harmonics. After expressing in terms of its Fourier coefficients and and in powers of , we use Eqs. (‎4.22)–(‎4.25) to compare powers of , which gives the following algebraic matrix-vector equation for :

|  |  |
| --- | --- |
| (‎5.54) |  |

with

|  |  |
| --- | --- |
| (‎5.55) |  |

|  |  |
| --- | --- |
| (‎5.56) |  |

where and the terms and are unknown and must be found. Consider the following decomposition of the matrix :

|  |  |
| --- | --- |
| (‎5.57) |  |

where

|  |  |
| --- | --- |
| (‎5.58) |  |

The function arranges the matrices into a block diagonal form (also known as a *direct sum*). Each 2×2 block on 's diagonal is a generalized “eigenvalue” that can be assigned to and corresponds to each column block in that is a generalized “eigenvector” that can be assigned to . In this particular case, each 2×2 block on 's diagonal is a matrix representation of the eigenvalues , (e.g., ). Since we have no prior information regarding [in contrast with , which can be obtained from the eigenvalues of ], we need to examine each block generalized eigenvalue and the corresponding generalized eigenvector. To simplify the notation, we use:

|  |  |
| --- | --- |
| (‎5.59) |  |

where is a generalized eigenvalue corresponding to a compatible generalized eigenvector. If we set and , then a potential solution for is:

|  |  |
| --- | --- |
| (‎5.60) |  |

It is easy to check that . In addition, by inserting our guess for into the RHS of Eq. (‎2.8), we can check whether the value for the constant part is valid:

|  |  |
| --- | --- |
| (‎5.61) |  |

We thus have a potential solution for the constant part , which shows that is not a function of . This solution for can be decomposed into the following sum:

|  |  |
| --- | --- |
| (‎5.62) |  |

Note that in this equation is identical to derived from Eqs. (‎5.58)–(‎5.60). In addition, it is easily verified that and are similar matrices because they share the same Jordan block matrix , which is similar to each matrix and individually.

To summarize, we conclude that a solution to construct the transition matrix of the matrix representing the LPTV system is formed from the matrices , where:

|  |  |
| --- | --- |
| (‎5.63) |  |

This solution is consistent with the data given in rows E of **Tables ‎3** and **4**.

#### **Example ‎5.5: Rows F of Tables ‎3 and 4**

Consider the following real 2×2 matrix for a LPTV system :

|  |  |
| --- | --- |
| (‎5.64) | , |

which has harmonics. By writing , we obtain:

|  |  |
| --- | --- |
| (‎5.65) |  |

Note that and , so it is not necessary to zero the trace. In addition, the matrix evaluated at ,

|  |  |
| --- | --- |
| (‎5.66) |  |

is decomposed as follows:

|  |  |
| --- | --- |
| (‎5.67) |  |

where

|  |  |
| --- | --- |
| (‎5.68) |  |

To recast the matrix, we use, to obtain

|  |  |
| --- | --- |
| (‎5.69) |  |

with

|  |  |
| --- | --- |
| (‎5.70) | . |

Note that is exactly the LPTV system matrix defined in rows G of **Tables ‎3** and **4**,and that and . We now solve for and, to simplify the problem, we omit the subscript [e.g., ].

We assume the transition matrix is decomposed according to *Floquet theory* into a periodic part and a constant part . Since has harmonics and is of size , we hypothesize that has harmonics. After expressing in terms of its Fourier coefficients and and powers of

|  |  |
| --- | --- |
| (‎5.71) |  |

with

|  |  |
| --- | --- |
| (‎5.72) |  |

|  |  |
| --- | --- |
| (‎5.73) |  |

where and the terms and are unknown and must be found. Note that has eigenvalues that can be represented by the 2×2 matrix

By taking

|  |  |
| --- | --- |
| (‎5.74) |  |

the terms and must be defined by the corresponding real and imaginary parts of the eigenvector , so and .

By setting

|  |  |
| --- | --- |
| (‎5.75) |  |

Eq. (‎5.72) is satisfied. Let and be the blocks for as follows:

|  |  |
| --- | --- |
| (‎5.76) |  |

This implies that:

|  |  |
| --- | --- |
| (‎5.77) |  |

Note that and , which indicates that our choice for is correct. Since can be defined up to a scale factor, we can, without loss of generality, multiply our solution for by −1/2 to normalize the result so that , i.e.,

|  |  |
| --- | --- |
| (‎5.78) |  |

We now insert our guess for into the RHS of Eq. (‎2.8) to check if the constant part is valid. The result is:

|  |  |
| --- | --- |
| (‎5.79) |  |

We thus conclude that the matrices form a solution to construct the transition matrix for the matrix of the LPTV system, with:

|  |  |
| --- | --- |
| (‎5.80) |  |

This solution is consistent with the data given in rows G of **Tables ‎3** and **4**. To calculate the transition matrix of the original example, we use Eq. (‎5.28), which gives:

|  |  |
| --- | --- |
| (‎5.81) | and |

Since is defined up to a scale factor, we can multiply by −1 so that our final result is the pair of matrices and for the LPTV system such that:

|  |  |
| --- | --- |
| (‎5.82) |  |

which is consistent with the data given in rows F and G of **Tables ‎3** and **4**.

∎

#### **Example ‎5.6: A 3×3 example**

Consider the following real matrix for a LPTV system:

|  |  |
| --- | --- |
| (‎5.83) |  |

=

+

+

+

where , , and and are -periodic 3×3 matrices () with five harmonics () given by the following Fourier series: For ,

|  |  |
| --- | --- |
| (‎5.84) | . |

For ,

|  |  |
| --- | --- |
| (‎5.85) | . |

For simplicity, we present all and with the common denominator of eight, so we did not simplify the fractions. Observe that and, in addition, evaluates at to:

|  |  |
| --- | --- |
| (‎5.86) |  |

This constant matrix has the following eigen decomposition:

|  |  |
| --- | --- |
| (‎5.87) | *,*  , |

where is the Jordan block corresponding to the eigenvalues , and is the corresponding matrix of eigenvectors of . We can find matrices and for [see Eq. (‎2.6)], where:

|  |  |
| --- | --- |
| (‎5.88) |  |

Note that multiplying from the right by and using the associative property of matrix multiplication gives:

|  |  |
| --- | --- |
| (‎5.89) |  |

We can find an equivalent pair for the solution, and . For , we take , , and . In addition, we must have and [and and ]. Since has harmonics and is of size , we hypothesize that the number of the harmonics of is . After expressing in terms of its Fourier coefficients and and in powers of ω, we use Eqs. (‎4.22)–(‎4.25) to compare powers of , obtaining the following algebraic matrix-vector equation for :

|  |  |
| --- | --- |
| (‎5.90) |  |

with

|  |  |
| --- | --- |
| (‎5.91) | , |

|  |  |
| --- | --- |
| (‎5.92) |  |

The matrix can be decomposed by using the following eigen decomposition:

|  |  |
| --- | --- |
| (‎5.93) | *,* |

where

|  |  |
| --- | --- |
| (‎5.94) |  |

|  |  |
| --- | --- |
| (‎5.95) |  |

We only consider eigenvectors corresponding to eigenvalues that are compatible with the Jordan block . If we set such that , then we have the following solution for :

|  |  |
| --- | --- |
| (‎5.96) | *,* |

which has the properties and so that we can take . We insert into the RHS of Eq. (‎2.8) to check if the constant part is valid; the result is:

|  |  |
| --- | --- |
| (‎5.97) |  |

The solution of the constant part is valid because it is constant (i.e., independent of ). We obtain [see Eq.(‎5.86)] and .

We conclude that the periodic part in Eq. (‎5.96) and the constant part in Eq. (‎5.97) are the solution matrices needed to construct the transition matrix of the matrix [Eq. (5.83)] of the LPTV system. Based on Eq. (‎5.89) (up to a scale factor for the transformation matrix ) we can use other matrices and as solutions for the LPTV system of Eq. (‎5.83):

|  |  |
| --- | --- |
| (‎5.98) | = |

|  |  |
| --- | --- |
| (‎5.99) |  |

∎

## Summary

This chapter assumes that , , and may be set up from a linear function in of the Fourier coefficients of . The Fourier coefficients of are constant (independent of ) and is a linear function of . By treating the frequency as a free parameter, we expand the algebraic equations for the Fourier coefficients of the LPTV system into powers of , which allows the coefficients to be matched to solve the system.

By setting in the matrix (under the assumption of continuity at ), we obtain a constant (time invariant) matrix that gives us information regarding . This information significantly reduces the computation of the matrix , so that we can choose to be any matrix that is similar to (in terms of matrix similarity). It is also possible to obtain from higher powers of (e.g., ), although greater computational power is required to select a compatible within the possible options generated by the eigen decomposition.

To find matrices and such that and , it is convenient to apply a similarity transform to the matrix of the LPTV system such that will have a Jordan canonical form [in addition to zero trace of ].

The bottleneck of the proposed approach is that we are limited to a finite number of harmonics for . In addition, the method requires more exploration to obtain the matrices for LPTV systems and their periodic part so that both have a finite number of harmonics. Once conditions are found that produce with a finite number of harmonics, the next step would be to generalize the structure of LPTV systems to solve them when and\or have infinite harmonics.

# Discussion

## Contribution of the Research

This work makes the following contributions to the study of LPTV systems:

1. **Decomposition into similar real-imaginary and even-odd Fourier series:** This work reveals a similarity between algebraic equations in the real-imaginary Fourier decomposition and the even-odd Fourier decomposition. Depending on the application, we can choose between the two methods to solve LPTV systems, although even-odd decomposition might prove more convenient to use on a real LPTV system.
2. **Stability analysis based on a variable frequency** : This allows us to define a condition on the frequency so that the LPTV system is stable, in contrast with previous works that rely on a fixed frequency.
3. **are polynomials in with a finite number of harmonics:** To address the case in which have afinite number of harmonics, the Fourier coefficients of
4. **Matrix similarity between and at** : At , is constant and similar to in the sense of matrix similarity. It might be convenient to transform to a new similar matrix such that the matrix has a real Jordan canonical form and to set . This transformation will reduce the required computational power since is already obtained. From we can obtain with the condition and then obtain directly, as discussed in the previous paragraph. In addition, it is convenient to zero the trace of so that, for all, and .
5. **The frequency may be varied for all** : In contrast with Yakubovich and Starzhinskii (Yakubovich & Starzhinskii, 1975), who use a perturbation analysis of a LPTV system with some small parameter , all analysis and solution procedures under the assumptions applied in this work (including continuity in ) are valid for all . Under these assumptions, is valid since (i.e., inserting before solving the ODE of the LPTV system or after solving it gives the same transition matrix). In addition, negative values of are valid because we can use the parity of cosine and sine functions to produce a new LPTV system reflected to positive frequencies.

## Suggestions for Future Research

We suggest studying the following issues for LPTV systems:

1. **Determine the conditions for such that both and have a finite number of harmonics:** To determine these condition, we may explore the following questions has harmonics, has *p* harmonics, and is the dimension of and ]:
   1. If , is ? Should we speculate ?
   2. If and . + another condition, then is ?
   3. Suppose that the Fourier coefficients of are polynomial in [e.g., ] such that we have . If we swap components among [e.g., +…] or swap sub-components among , with , does it imply ? Under which conditions is this true?
2. **Generalize the structures of :** We suggest exploring other possible structures of such that the procedure of solving the ODEs of LPTV systems by comparing powers of remains valid. For example,
   1. The Fourier coefficients of are polynomial in .
   2. The denominators of the components are parameterized with some arbitrary constant parameter or are polynomial in . Multiplying by the common denominator may be useful to obtain the solution.
   3. and\or can be presented as a quotient of a finite Fourier series in numerators of the matrix divided by a finite Fourier series in the denominators. Multiplying by the common denominator may be useful to obtain the solution.
3. **Studying LPTV systems under number systems other than and :** In this work, we focus on real LPTV systems [i.e., ]. In addition, we cover complex LPTV systems [i.e., ] by using a real 2×2 matrix representation . We suggest studying LPTV systems in different number systems, such as dual numbers, split complex numbers, or any variation thereof [see, e.g., (Akar, Yüce, & Şahin, 2018), (Dattoli, Licciardi, Pidatella, & Sabia, 2018)].
   1. Split complex numbers are also known as hyperbolic complex numbers . In special cases, it is possible to use these in even-odd decomposition (see ‎APPENDIX C). Note that .
   2. Use dual numbers or any isomorphic numbers system [see, e.g., (Klein & Maimon, 2019) Section 6.2]. This number system might be useful for a small-perturbation approach, as suggested by (Yakubovich & Starzhinskii, 1975) in Chapter 4. Note that .
4. **Extend the notion of LTI tools in LPTV systems (Wereley, 1991):** In his Ph.D. thesis (Wereley, 1991), Wereley focuses mainly on defining the Toeplitz transform, harmonic transfer functions, zeros and poles, etc., which are related to the exponential representation of infinite Fourier series for LPTV systems and are especially used with the Hill Equation. I suggest extending these definitions for use in a cosine-sine representation of Fourier series and to explore cases with a finite number of harmonics.

# Fourier Series for Matrices

Suppose that matrix is a periodic function with period or, equivalently, with a frequency . In this case, can be decomposed into a Fourier series. In this work, we use the finite summation version of the Fourier series decomposition to focus on cases in which the LPTV system matrix and its transition matrix involve a finite Fourier series decomposition [suppose has harmonics, then take for the infinite version of the Fourier series].

Traditionally, there are two forms of a Fourier series decomposition:

The exponential form is given by:

|  |  |
| --- | --- |
| (‎A.1) |  |

where is the imaginary unit and is the Fourier series coefficient and is computed by:

|  |  |
| --- | --- |
| (‎A.2) |  |

If is real, then .

The cosine-sine form is given by:

|  |  |
| --- | --- |
| (‎A.3) |  |

where

|  |  |
| --- | --- |
| (‎A.4) |  |

for . We can expand the sum in Eq. (‎A.3) by exploiting the following properties of even-odd symmetry from Eq. (‎A.4):

|  |  |
| --- | --- |
| (‎A.5) |  |

By using these properties, we rewrite Eq. (‎A.3) as follows:

|  |  |
| --- | --- |
| (‎A.6) |  |

We use the exponential form and the cosine-sine form for different purposes: The exponential form, due to its compactness, is used to define or prove general properties when neither a real analysis nor the even-odd properties are involved. The cosine-sine form is used to define or prove properties when either a real analysis or the even-odd properties are involved (e.g., to analyze the transition matrix when the system matrix is real).

**Lemma ‎A.1**: If and are square matrices (with the same size and time period), such that has harmonics and has harmonics, then has at most harmonics*.*

**Proof:** Let and , then

From the equation above, we see that the Fourier coefficient of is given by the sum for all such that . Note that some Fourier coefficients may be zero, so the number of the harmonics of is at most ∎.

**Lemma ‎A.2:** For any and for any matrix , if has a finite number of harmonics, then has *at most* harmonics.

**Proof:** We use proof by induction, for ( is fixed). For we have (for harmonics, the result is trivially obtained), therefore we start our base case by .

Base Case:

If has harmonics, then

has *at most* harmonics because each element has at most harmonics, so any product of two elements can produce a term with *at most* harmonics (and, therefore, any linear combination of this product). We conclude that the lemma is true for the base case ().

Induction Hypothesis: Assume the lemma is true for .

Note that, for , we have

for fixed ,

where is the minor matrix of generated by removing row and column (therefore, ).

Assume that any with harmonics, has *at most* harmonics.

Induction Step: Prove that the hypothesis holds for .

Note that, for we have

for fixed .

In this case, . By the induction hypothesis, we assume that all minors of , in for all , obtain the property of having *at most* harmonics. Since each element has at most harmonics, then each product (and any linear combination thereof) has *at most* harmonics.

Conclusion: We have proven the lemma for the base case ( or for the trivial case). The validity of the assumption of the induction hypothesis () implies that the induction step () is true for any arbitrary . Therefore, by induction, the lemma is proven.

∎.

**Lemma ‎A.3:** For any and for any matrix , if has a finite number of harmonics, then the adjoint has *at most* harmonics.

**Proof:** The adjoint is defined as the transpose of the cofactor matrix of , which is constructed from the minor elements , i.e.,

Referring to **Lemma A.2**, each determinant above has *at most* harmonics, so the matrix adjoint has *at most* harmonics.∎

**Observations**:

1. We have consistently stated above “*at most* harmonics” because the exact number of the harmonics might be less than expected. For example, consider an periodic matrix with harmonics: it is possible that the number of harmonics given by is or even zero.
2. If , then has *at most* harmonics since and has *at most* harmonics.
3. If (in addition to item 2) , then has *exactly*  harmonics because, for a matrix, no multiplication is involved to compute the adjoint, only a change of position or sign; i.e.,   
   .

# Exponential Fourier Analysis for LPTV Systems

## General

Suppose that the periodic matrix and are the pair of matrices that solve Eq. (‎2.6) and construct the transition matrix . Consider the following exponential Fourier series decomposition for and :

|  |  |
| --- | --- |
| (‎B.1) |  |

|  |  |
| --- | --- |
| (‎B.2) |  |

where and are the complex Fourier coefficients of for the matrices and , respectively. Inserting the above definition into Eq. (‎2.6), re-indexing the LHS of that equation by , and collecting factors of gives:

|  |  |
| --- | --- |
| (‎B.3) |  |

Given that Eq. (B.3) holds for all , by comparring coefficients of , we obtain the following set of algebraic equations in and :

|  |  |
| --- | --- |
| (‎B.4) |  |

Equivalently, by re-indexing the LHS, we obtain

|  |  |
| --- | --- |
| (‎B.5) |  |

which can be represented as an infinite block-system of equations

|  |  |
| --- | --- |
| (‎B.6) |  |

where

|  |  |
| --- | --- |
| (‎B.7) |  |

and

|  |  |
| --- | --- |
| (‎B.8) |  |

Based on (Wereley, 1991), we decompose into the difference between a *Toeplitz transform* of from a block diagonal form of the terms so that:

|  |  |
| --- | --- |
| (‎B.9) |  |

where is the *Toeplitz transform* operator, which maps an infinite exponential Fourier series with matrix coefficients into an infinite block matrix.

In general, the LPTV system may be a complex system [i.e., is a complex matrix], so and might also be complex. It is also possible that and are complex even though is real. However, we limit ourselves in this work to finding procedures that obtain real constant matrices from real LPTV systems and to using properties of Fourier series applicable to real functions (e.g., ).

## Real-Imaginary Decomposition

In the following real-imaginary decomposition, we d as:

|  |  |
| --- | --- |
| (‎B.10) |  |

|  |  |
| --- | --- |
| (‎B.11) |  |

Using the assumption that and are real, we have:

|  |  |
| --- | --- |
| (‎B.12) |  |

|  |  |
| --- | --- |
| (‎B.13) |  |

From the property of the imaginary part, we conclude that . Inserting Eqs. (‎B.12) and (‎B.13) into Eq. (‎B.4) and comparing the real and imaginary parts gives the following set of algebraic equations in , , and :

|  |  |
| --- | --- |
| (‎B.14) |  |

Equivalently, by re-indexing the LHS of Eq. (B.14) [cf. Eq. (‎B.5)],

|  |  |
| --- | --- |
| (‎B.15) |  |

To reduce the negative indices of and , we integrate the properties of Eqs. (‎B.12) and (‎B.13) into Eq. (‎B.15) and re-index the LHS of Eq. (‎B.15) to collect factors of and for . The result is:

|  |  |
| --- | --- |
| (‎B.16) |  |

for ,….

For , we have:

|  |  |
| --- | --- |
| (‎B.17) |  |

so equation [of Eqs. (B.17)] is identically zero due to its real matrices and :

* ;
* ;
* ;
* .

Equation (‎B.16) can be represented as a semi-infinite block-system of the equation

|  |  |
| --- | --- |
| (‎B.18) |  |

where

|  |  |
| --- | --- |
| (‎B.19) |  |

|  |  |
| --- | --- |
| (‎B.20) | = |

|  |  |
| --- | --- |
| (‎B.21) |  |

is the block matrix in row and column that multiplies the element , and is the Kronecker delta function (={0 for , 1 for }).

# Representing LPTV Systems by 2×2 Real Blocks and Split Complex Numbers

## Representing Complex LPTV System by 2×2 Real Blocks

In general, a LPTV system may be complex, which means that the system matrix is complex, so at least one of the matrices and is complex. In this case, the real-imaginary decomposition equation and the properties discussed in Section ‎B.2 are irrelevant because the assumptions that , , and are real are not valid. We consider two approaches.

First, we search the complex plane for a solution to the LPTV system as described in Section ‎B.1. Alternatively, we convert the complex LPTV system into a LPTV system with real and imaginary parts separated and rearrange the system into 2×2 real block matrices. This approach is based on representing every complex number by a 2×2 real block matrix , as described below.

We use the following notation:

* ;
* ;
* ;
* .

The new notation is defined such that and . Using the new notation in Eq. (‎2.1) to obtain the new complex LPTV system in terms of gives:

|  |  |
| --- | --- |
| (‎C.1) |  |

For Eq. (‎2.6), we use *Floquet theory* to obtain the complex linear differential equation in terms of the matrices and . The result is:

|  |  |
| --- | --- |
| (‎C.2) | ⇒ |

By separating the real and imaginary parts and rearranging the results into 2×2 real block matrices, we obtain a real LPTV system equivalent to that given in Eq. (‎C.1):

|  |  |
| --- | --- |
| (‎C.3) |  |

As per *Floquet theory*, this is solved by the following real linear differential equation equivalent to Eq. (‎C.2):

|  |  |
| --- | --- |
| (‎C.4) |  |

We denote the state as , the real LPTV system as , and the *Floquet theory* real matrices as and . To fit the dimensions of the matrices and solve for and , we construct every block-matrix from Eq. (‎C.2) to (‎C.4) through isomorphic representations of every complex number by a 2×2 real block matrix (). However, to solve for the state , it suffices to arrange it as a column vector . Note that, if a real LPTV system matrix has the form , then, by considering the equation backwards, we conclude that its *Floquet theory* real matrices have the same form: and , so we reform the LPTV system into the complex plane as described in Eq. (‎C.2). As a result, we obtain some of the properties described previously in this work.

## Representing Even-Odd Decomposition of LPTV Systems by 2×2 Blocks

This section follows the ideas of Section ‎4.2.2 and uses an even-odd decomposition, but with the state vector separated into even and odd parts. Furthermore, the matrix of the LPTV system is separated according to blocks. We rewrite Eqs. (‎4.12) and (‎4.13) in the following block matrix form:

|  |  |
| --- | --- |
| (‎C.5) |  |

which, by *Floquet theory*, is related to the following LPTV system:

|  |  |
| --- | --- |
| (‎C.6) |  |

The form in Eq. (‎C.6) can also be derived from an even-odd decomposition of Eq. (‎2.1):

|  |  |
| --- | --- |
| (‎C.7) |  |

As in the previous section C.1, we denote the state as , the real LPTV system as , and the *Floquet theory* real matrices as and . In addition, we observe that, if a LPTV system matrix has the form , then, by looking at the equation backwards, we conclude that its *Floquet theory* real matrices have the same form: and . We recast this LPTV system into a more compact form, as described by Eq. (‎4.11), so we obtain some of the properties previously described in this work.

For example, referring to Eqs. (‎2.4) and (‎2.5), we note the average and trace properties for and : and : Average , , Trace: , .

We generalize the structure of the even-odd decomposition by using, e.g., split complex numbers (also known as hyperbolic complex numbers) , which are defined as follows:

|  |  |
| --- | --- |
| (‎C.8) |  |

If we replace the imaginary unit (where ) by the split imaginary unit (with ) in Eqs. (‎C.1) and (‎C.2), we obtain:

|  |  |
| --- | --- |
| (‎C.9) |  |

|  |  |
| --- | --- |
| (‎C.10) | ⇒ |

That can be represented block-wise by:

|  |  |
| --- | --- |
| (‎C.11) |  |

|  |  |
| --- | --- |
| (‎C.12) |  |

from which, with a suitable change of notation, we find that the even-odd decomposition in Eqs. (‎C.6) and (‎C.5) is a special case of Eqs. (‎C.11) and (‎C.12), respectively.

# Dynamic Eigen Decomposition of LTV Systems

This appendix outlines the main results of (Wang, 2017), who generalizes the notion of eigen decomposition of LTV systems [the related proofs are given in (Wang, 2017)]. Consider the following general LTV system defined by the matrix :

|  |  |
| --- | --- |
| (‎D.1) |  |

According to (Wang, 2017), a LTV system matrix can be decomposed as follows:

|  |  |
| --- | --- |
| (‎D.2) |  |

where is the *k*th dynamic (-variant) eigenvalue of , is the corresponding dynamic (column) eigenvector, is the row eigenvector, and the reciprocal of such that is the Kronecker delta function. The pair is called the eigenpair of . The transition matrix is constructed from the eigenpairs of as follows:

|  |  |
| --- | --- |
| (‎D.3) |  |

Each eigenpair is a solution to obtain the pair in the following equation:

|  |  |
| --- | --- |
| (‎D.4) |  |

The procedure suggested in (Wang, 2017) is to use an *auxiliary equation*; that is, to define a new LTV system with an arbitrary matrix such that:

|  |  |
| --- | --- |
| (‎D.5) |  |

Equation (‎D.4) can be rearranged as:

|  |  |
| --- | --- |
| (‎D.6) |  |

which has a nontrivial (nonzero) solution for if and only if:

|  |  |
| --- | --- |
| (‎D.7) |  |

The procedure consists of the following steps: Select an arbitrary matrix ; solve Eq. (‎D.7) to find all eigenvalues ; solve Eq. (‎D.6) to find all eigenvectors ; if Eq. (‎D.5) is *not* satisfied, select a new arbitrary matrix and repeat the process; if Eq. (‎D.5) *is* satisfied, construct [refer to Eq. (‎D.3)].

**Remarks:**

1. The set of all eigenpairs are not a unique solution [depends on the selection of ].
2. If satisfies the commutative property, , then we can use , which implies that each eigenvector is a constant vector.
3. If , then the eigenvalues are the result of the Riccati equation , with the corresponding eigenvectors . A more general procedure for using the Riccati equation is given by van der Kloet and Neerhoff (van der Kloet & Neerhoff, 2004a), (van der Kloet & Neerhoff, 2004b).

# Generalization of LTI-System Tools for LPTV Systems

This appendix outlines the main results of (Wereley, 1991), which suggest generalizing tools for LPTV systems to analyze and control LTI systems (e.g., transfer functions, zeros and poles in and domains). In this appendix, we focus only on the *s*-plane generalizations. Consider the following state space for a LPTV system:

|  |  |
| --- | --- |
| (‎E.1) |  |

where each matrix set is -periodic with exponential Fourier series representation:

|  |  |
| --- | --- |
| (‎E.2) |  |

Consider for some and let be the following input:

|  |  |
| --- | --- |
| (‎E.3) |  |

According to (Wereley, 1991), theorem 3.14 for zero initial condition and the steady state, , , and have the same form as :

|  |  |
| --- | --- |
| (‎E.4) |  |

Upon inserting Eqs. (‎E.2)–(‎E.4) into Eq. (‎E.1), re-indexing the -periodic matrix set by, e.g., , comparing the coefficients of , and using , where is the Kronecker delta function, we obtain:

|  |  |
| --- | --- |
| (‎E.5) |  |

Based on (Wereley, 1991), theorem 3.14 and on the *Lyapunov reducibility theorem* [see (Yakubovich & Starzhinskii, 1975), Chapter 2, Section 2.4], and assuming that we know the matrices [or at least ][[6]](#footnote-7) to obtain the transition matrix , we use to transform Eq. (‎E.1) into a state-space model with time-invariant dynamics in the following sense:

|  |  |
| --- | --- |
| (‎E.6) |  |

where (see footnote7), , , .

**Definition ‎E.1**: The *harmonic state-space model* is defined by the system of equations (‎E.5) represented by:

|  |  |
| --- | --- |
| (‎E.7) |  |

where are the Toeplitz transforms of , respectively [see Eq. (‎B.9)], , and , are the doubly infinite vector representations of , respectively (e.g., ). The set is used to denote the harmonic state-space model. If the state-space model is transformed so that it has time-invariant dynamics [Eq. (‎E.6)], then the set of corresponding Toeplitz-transformed matrices is used, where , ,, and .□

**In the s plane, the form of a LPTV *harmonic state space* is similar to that of a LTI *state space*.**

**Definition ‎E.2**: The *harmonic transfer function* is an infinite-dimensional matrix that describes how the input is related to the output such that:

|  |  |
| --- | --- |
| (‎E.8) |  |

where is the infinite identity matrix, if the inverse exists. If the state-space model is transformed to have a time-invariant dynamic [Eq. (‎E.6)], is identically obtained by:

|  |  |
| --- | --- |
| (‎E.9) |  |

**In the s plane, the form of a LPTV *harmonic transfer function* is similar to that of a LTI *transfer function*.**

**□**

**Definition ‎E.3**: The *LPTV poles in the plane* mark the location in the complex plane where the harmonic transfer function is not analytic. □

The poles of a LPTV system are obtained by solving the eigenvalue problem:

|  |  |
| --- | --- |
| (‎E.10) |  |

where, for each eigenvalue, is a pole of the LPTV system with a corresponding infinite eigenvector . Equivalently, if the LPTV state space is transformed to be time invariant [Eq. (‎E.6)], the poles can be obtained from the union of all the eigenvalues from the eigenvalue problem:

|  |  |
| --- | --- |
| (‎E.11) |  |

**In the s plane, the form of a LPTV pole definition is similar to that of a LTI pole definition.**

**Definition ‎E.4**: The *LPTV transmission zeroes in the plane* mark the location in the complex plane, along with the corresponding input:

|  |  |
| --- | --- |
| (‎E.12) |  |

The initial condition for state implies that the output is identically zero, i.e., . □

These transition zeroes can be obtained by solving the following generalized eigenvalue problem:

|  |  |
| --- | --- |
| (‎E.13) |  |

where contains the harmonics determined for , and contains the harmonics for determined by the initial condition . Equivalently, if the LPTV state space is transformed to be time invariant [Eq. (‎E.6)], the transmission zeroes are obtained by solving the eigenvalue problem

|  |  |
| --- | --- |
| (‎E.14) |  |

**In the s plane, the definition of the transmission zeros for a LPTV system is similar to that of a LTI system**.

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***מילות מפתח*:** תיאוריית פלוקה; משוואות דיפרנציאליות לינאריות; מערכות לינאריות משתנות בזמן; מערכות מחזוריות; טורי פורייה; משוואות מטריציות; פירוק ספקטרלי; ניתוח במישור התדר; השוואה מקדמי חזקות; מספר הרמוניות סופי

**תקציר**

עבודה זו מציגה ניתוח של מערכות לינאריות מחזוריות המיוצגות על ידי מערכת משוואות דיפרנציאליות לינאריות רגילות בארגומנט הזמן עם מטריצת מקדמים מחזורית עם זמן מחזור כלשהו (או באופן שקול בעלת תדירות כלשהי) המבוססת על תיאוריית Floquet. לפי תיאוריית Floquet עבור מערכת לינארית מחזוריות המיוצגת ע"י מטריצה ריבועית , מטריצת מעבר של ניתנת לייצוג ע"י מכפלה של מטריצה ריבועית מחזורית באקספוננט של מטריצה מהצורה כאשר הינה מטריצה קבועה ביחס ל-. למרות נכונות תאוריה זו, קשה מאוד למצוא בצורה אנליטית ביטויים סגורים ל- ו-במקרים בהם מטריצת מעבר של לא יודעה מראש, או במילים אחרות קשה מאוד למצוא בצורה אנליטית ביטוי סגור לפתרון של מערכת מחזורית.

המטרות של העבודה הינה לבחון את ההשפעה של שינוי פרמטר התדר במערכות LPTV ופתרונן (לדוגמא: ניתוח יציבות), ולנסות לאפיין משפחה (רצוי גדולה) של מטריצות מחזוריות עם מספר הרמוניות סופי, אשר מתכן מתקבל כי החלק המחזורי של הפתרון הינו בעל מספר הרמוניות סופי. בנוסף, עבור משפחה זו נדרש למצוא דרך יעילה שתניב את ו-הדרושים לפתרון. שיטת המחקר הינה לבחון דוגמאות למטריצות מחזוריות אשר התדירות שלה הינו פרמטר חופשי (כלומר אינו נתון ע"י מספר קבוע), על מנת לקבל משפחה כמה שיותר גדולה של מערכות מחזוריות אשר למעשה מגדירות את מטריצת המעבר שלה כתלות ב- . בעבודה זו נתמקד במשפחה של מטריצות מחזוריות אשר יש להן פירוק פורייה סופי עם מקדמי פורייה שהינם פולינום סופי ב-, כאשר החלק המחזורי של הפתרון בעל פירוק פורייה סופי עם מקדמי פורייה בלתי תלויים בתדר והחלק הקבוע -הינו פולינום סופי ב-.

תוצאות המחקר מראות כי עבור המשפחה הנ"ל ניתן לבצע השוואת מקדמים לפי חזקות של בנוסף להשוואות הרמוניות (מקדמים של cos\sin או של exp מרוכב) על מנת לקבוע את הצמד את ו-. בנוסף על כך, מעבודה זו ניתן להסיק על הקשר בין מערכת לינארית מחזורית עם תדר כלשהו לבין מערכת לינארית בלתי תלויה בזמן המתקבלת כתוצאה מהצבה . בהנחת מבנה - כפולינום ב-,ניתן לבחור את איבר החופשי בפולינום להיות כל מטריצה הדומה למטריצה הקבועה של המחושבת ב- . חשיבות לשימוש ברעיון בו התדר הינו פרמטר חופשי הינה לבחינת היציבות של הפתרון כתלות בפרמטר אשר נובעת מתלות הערכים העצמיים של החלק הקבוע בפרמטר (כאשר אכך ישנה תלות כזו).

**אוניברסיטת בן-גוריון בנגב**

**הפקולטה למדעי ההנדסה**

**בית הספר להנדסת חשמל ומחשבים**

**המחלקה להנדסת חשמל ומחשבים**



**ניתוח של מערכות לינאריות, משתנות בזמן ומחזוריות**

חיבור זה מהווה חלק מהדרישות לקבלת תואר מגיסטר בהנדסה

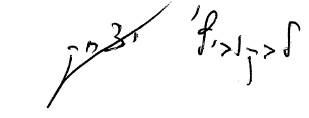
מאת: אורן פייבל

מנחה:

פרופ' יצחק לבקוביץ'



המחבר: אורן פייבל ............................. תאריך: ‏26 אוקטובר 2020



מנחה: פרופ' יצחק לבקוביץ' ............................. תאריך: ‏26 אוקטובר 2020

יו"ר ועדת הוראה לתואר שני:

שם: .......................................................... תאריך: .........................

**אוניברסיטת בן-גוריון בנגב**

**הפקולטה למדעי ההנדסה**

**בית הספר להנדסת חשמל ומחשבים**

**המחלקה להנדסת חשמל ומחשבים**



**ניתוח של מערכות לינאריות, משתנות בזמן ומחזוריות**

חיבור זה מהווה חלק מהדרישות לקבלת תואר מגיסטר בהנדסה

מאת: אורן פייבל

מנחה:

פרופ' יצחק לבקוביץ'

1. It may be referred to as the “Vinograd Example” (Vinograd, 1952) in the literature. [↑](#footnote-ref-2)
2. We limit our work to a real LPTV system matrix , so we would like to find a real matrix pair and . Therefore, we use a cosine-sine Fourier series to avoid complex matrix computations. In addition, we extend the Fourier series to negative indices (see ‎APPENDIX A for details). [↑](#footnote-ref-3)
3. Although this work is not limited to a small frequency parameter [in contrast to (Yakubovich & Starzhinskii, 1975) who define a small perturbation by ], this work discusses properties related to the LPTV system matrix and its LTI version when , so that the matrix is not periodic and its components become constant (i.e., is a LTI system). [↑](#footnote-ref-4)
4. Assuming that is defined, the antiderivative of may be defined by ; the antiderivative can thus be defined up to a constant offset. [↑](#footnote-ref-5)
5. Elementary functions may refer, e.g., to sums, differences, products, quotients, powers, exponentiations, or logarithms of hyperbolic or trigonometric function. However, piecewise functions, such as rectangular or sawtooth functions are beyond the scope of this chapter. [↑](#footnote-ref-6)
6. If is correct, then is obtained by so the RHS is constant matrix.

   If is incorrect, then is some -periodic matrix. [↑](#footnote-ref-7)