```latex
\documentclass[11.5pt]{amsart}
\usepackage{amssymb, latexsym, amsthm} %, makeidx}

%ALE: Edit this file
%\usepackage[margin=1.3in]{geometry}
\usepackage{xcolor}                  % For including pictures
\usepackage{pstricks-add}
\usepackage{tikz,pgfplots}
\pgfplotsset{compat=1.15}
\usetikzlibrary{arrows}
\usetikzlibrary[patterns]
\usepackage{cite}
\usepackage[colorlinks=true, urlcolor=blue, citecolor=black,
linkcolor=black]{hyperref}
\usepackage{graphicx,caption,subcaption}
%\usepackage[onehalfspacing]{setspace}
\usepackage{natbib}
\usepackage{setspace}


%\voffset=-30pt%-78pt
%\oddsidemargin = -30pt
%\evensidemargin= -30pt
%\marginparwidth = -30pt
%\textwidth = 495pt%486pt
\textheight = 550pt
\linespread{1.5}
\usepackage{natbib}


\usepackage{graphicx}
\theoremstyle{plain}% Theorem-like structures provided by amsthm.sty
\newtheorem{theorem}{Theorem}[section]
\newtheorem{lemma}[theorem]{Lemma}
\newtheorem{corollary}[theorem]{Corollary}
\newtheorem{proposition}[theorem]{Proposition}

\theoremstyle{definition}
\newtheorem{definition}[theorem]{Definition}
\newtheorem{example}[theorem]{Example}

\theoremstyle{remark}
\newtheorem{remark}{Remark}
\newtheorem{notation}{Notation}


\newif\ifXY % turns XY version on/off
\XYtrue     % Turn it on
%\XYfalse    % Turn it off
%
\ifXY
```

```
%\input xy
%\input xyidioms.tex
%\usepackage{xy}
%\xyoption{all} %
%\fi % For \ifXY

%\newcommand\ideal[1]{{\left<#1\right>}}\begin{document}
%\thispagestyle{empty}
% \ \\
%\fontsize{14bp}{14pt}\selectfont
%\begin{center}{Full project description}\end{center}
\pagenumbering{gobble}

%\vspace{.26in}
\begin{document}
\begin{flushright}{Application No. 1116/24} \\PI1 Name: Shai Gul
\end{flushright}
\section{Scientific background}
%\textcolor{blue}{why is this research can be a tool in education???I edited
the text hope now it is better}
 Mathematics is key to many fields and is relevant to the vast majority of
tertiary students. Yet most students shy away from this discipline, viewing
it as a field that merely deals with quantities. Beyond being a factor in
student undergraduate course choice, this bias also limits the ability of
students to realize the full potential of the fields in which they have
chosen to major. In many faculties, students are not aware of what
mathematics has to offer---that it indeed deals with quantities but also with
patterns, structures, changes and space.

Perhaps the greatest lack of undergraduate math concepts can be found in the
faculties of art and design, which usually do not include mathematical ideas
in their curriculum (~~except ~~other than basic geometry).
 Designers, ~~and ~~especially industrial designers, are educated in ~~the
academy~~academia to innovate new products and features. They are driven by
this objective to push their boundaries with the help of other scientific
domains, including materials engineering (especially mockups in ~~3d ~~3D
printing), artificial intelligence, mechanics, and other fields. From my
point of view, mathematics has a variety of tools (algebra, topology, etc.)
that are just waiting for the right open-~~-~~minded designer to ~~be applied~~apply
them~~; algebra, topology, etc.~~. Theoretical m~~M~~athematical ~~theoretical ~~tools
~~may ~~can be considered not only ~~by ~~for patterns in the finishing process of a
given product but also in the initial steps of planning a product. In some ~~of
the ~~cases, questions such as ``Is it possible to define a product $X$ with
properties $Y$?'' can be answered in the planning process ~~by ~~using
mathematical justifications, as ~~have been ~~described in
\cite{bridges2023:413}.


 ~~In t~~This propos~~ition, we~~al will focus on both the symbiotic relationship
between mathematics and design~~,~~ and how the tools of each of these distinct
fields can ~~provide ~~lead to scientific innovations ~~to ~~in the other.



 % Yet, as our recent studies show, visualizing math concepts, such as
theorems, can open the world of math to undergraduates and make them more
open to adopting its many powerful tools.
```

% Conversely, visualizing elements not typically associated with math, such as music and art, can be used as a strategy to attract undergraduates in such fields to explore mathematical approaches. I here propose to validate this concept by demonstrating how utilizing math-based visualization can make math more accessible to non-math students and especially design students.
My vision is to propose and establish a field ~~, In a~~ similar ~~way~~ to mathematical~~-~~ physics~~,~~ and mathematical~~-~~ biology ~~, to provoke and establish the~~called \emph{mathematical~~-~~ design} ~~field~~ and show how important it is that intermediate math ~~will be~~is a part of the curriculum in different ~~studies~~ courses ~~of~~ on art and design.

% I believe that exposing students in non-math disciplines to math-visualization approaches can help them develop an interest in other, non-quantitative mathematics, such as topology and algebra. Moreover, even the average candidate for a mathematics undergraduate degree does not really know what to expect from the track, and I believe that if the average candidate knew what the field had to offer, these departments would become much more attractive. This could be achieved by implementing multidisciplinary tools, such as 3D visualisation and industrial tools, that can enliven mathematical conceptual ideas.

% \begin{figure}[htp]
% \begin{center}
% %\hspace*{-2cm}
% \includegraphics[scale=0.3]{klein_long_zip.jpg}
% \end{center}
% \vskip -0.1in
% \caption{One of the described cases in \cite{shahar}: A surface homoeomorphic to a Klein bottle. In the first three images, a cross-cap is zipped, leaving a gap. Since the short zippers now face each other in opposite directions, one of them must be flipped to form the handle. In the fourth image, the "bottom" short zipper is pulled through the gap in the long zipper to meet the "top" short zipper.}
% \label{fig:zip_klein2}
% \end{figure}

\section{Objectives and significance}
The main objective of this project is to connect mathematics, art, and design. We will show how mathematics with computational tools can define innovation in design and art, and more surprisingly, how design concepts can inspire ~~to define~~the development of new mathematical ideas. In this ~~proposition~~proposal, the focus is o~~i~~n the following three topics:
\begin{enumerate}
\item \textbf{Aim 1: ~~Classifying and defining~~Definition and classification of songs as three-dimensional (3D) objects ~~(Aim 1)~~}. Can a given song in Western music be modeled as a collection of curves or~~,~~ surfaces, or even be defined as a tangible object? If they indeed can be modeled in a pure mathematical fashion, can we sort songs ~~by~~ using equivalence relations? This research involves music, industrial design, differential geometry, algebra, and topology.

\item \textbf{Aim 2: Gradient topology ~~(Aim 2)~~}.

Commented [K1]: I suggest this heavy edit to ensure like is compared with like (fields are compared with other fields). The emphasis is optional but I think it will add to the text.

Commented [K2]: In some places, 3D is used, but in others, “three dimensional” is used. My recommendation is to define “3D” here on first use and use it everywhere else afterwards.

```
 A g~~G~~radient is an important concept in mathematics, and surprisingly, this
concept is also well defined from a designer's point of view as a ~~soft~~
gradual change in color~~-changing~~ (which contains the mathematical definition)
in a given image.
  \begin{figure}[htp]
\begin{center}
%\hspace*{-2cm}
\includegraphics[scale=0.2]{Figures/tmp.png}
\end{center}
\vskip -0.1in
\caption{A classical visual gradient in a grayscale color space. ~~In t~~This
case can be considered ~~as ~~to be the fundamental polygon of a cylinder, since
the upper ~~side ~~and ~~the ~~lower sides are in the same color direction.}
\label{fig:basic_gradient}
\end{figure}
%  \begin{figure}[htp]
% \centering
% %\vskip -0.8in
%   \includegraphics[width=0.4\textwidth]{Figures/tmp.png}
%    %\vskip -0.8in
% %Images/
% \caption{A gradient which is defined in the gray color space. In this case
can be considered as the oriented square of a cylinder.}
%   \hspace*{2cm}
%   \vskip -0.1in
%  \label{fig:cylider}
% \end{figure}
A~~s a mathematician~~lthough this simple design~~-~~ gradient, as shown in
Fig.~\ref{fig:basic_gradient}, ~~although it ~~has a geometrical property, it
reminds us as ~~us ~~mathematicians of the construction of a cylinder from the
respective fundamental polygon, which is obtained by attaching the edges with
identical ~~edges~~colors. This observation led us to ~~think is it~~wonder if it is
possible to define design-gradients for different topological surfaces
(toruses, Klein bottles, etc.), and if the answer is yes, ~~can ~~whether we ~~give~~
assign an upper and lower bound to the number of ~~each ~~gradients that exist~~s~~
for each topological surface. This research involves design, topology,
combinatorics, and complexity; ~~and ~~all ~~is ~~are influenced by design concepts.

\item \textbf{Aim 3: Defining dynamical tiling~~'s~~ in industrial design~~ (Aim
3)~~}.
It ~~turns out~~has been shown that algebraic structures can help designers in
the planning ~~steps ~~stages to ~~know ~~determine if a dynamical transformation of
the components can be obtained~~,~~ ~~all ~~simply by using a respective mechanism
~~which ~~that defines movement between different patterns/arrangements, each of
which accomplishes a different goal, as ~~has been ~~introduced in
\cite{bridges2023:413,Fletcher}. ~~It ~~This approach can be applied to folding
tables (reduction and expansions), lightning systems (exposure and
~~hiding~~concealment), and ~~more~~other design tasks.
We~~I~~ intend to generalize this result not only for planar patterns but also
for spherical patterns and especially ~~for ~~geodesic domes. This research
involves industrial design (~~three-dimensional~~3D visualization and
~~mechanicsm~~mechanism construction), differential geometry, and groups.
\end{enumerate}
%The second part of this proposal aims to use new platforms to engage
graduate mathematics in a new way. Mathematics, is usually have a very
conservative teaching approach, where the students need to ``train'' enough
(homework) to understand the main idea of a given course or subject. To my
```

Commented [K3]: Please check if this is your meaning. For general readers, I think this will be easier to understand if it is the correct interpretation. If not, it is fine to go back to the original.

opinion (and I think for most of the students), this kind of learning feels dry and might lead to a superficial understanding. I would like to change this experience to the students by making the course much more playful. If mathematics's could playful as the popular game `minecraft' i do not think we had to convince how import it to solve homework. In [DAVID AND SHAI], a playful game has been constructed: ``The symmetriods'', game, which is simply an arcade game that is based on the original Atari asteroid games that was popular in arcade machines in the late 70s and early 80s. The player chose a surface (torus, Klein bottle or M{\"o}bius strip) and need to decide which symmetry patterns are intruding to the surface. The intruders are defined as the elements which are not homeomorphic the chosen surface. It teach the player classification of surfaces and ideas and group theory. In this proposal we intend to create a game which we hope replace the traditional homework by just playing a game.

\section{About the lab}
  The Lab for Designing Mathematics, which I head, is a multidisciplinary research lab focused on ideas that involve advanced mathematics and design (especially industrial design). My team uses diverse tools from various faculties to achieve this aim, from computer science to industrial design. To advance our goal, we collaborate with various departments on campus, such as design (first and second degrees), computer science (first and second degrees), and applied mathematics~~;~~ , ~~and~~ as well as other institutes.
  The team aims to connect research fields that are traditionally perceived as starkly different, e.g., math and~~,~~ design, including music and art design. We are driven by the belief that our efforts can aid in the dissemination of intermediate mathematics concepts among designers and artists and, of course, help apply non-trivial mathematical ideas not only in traditionally ~~connections~~ connected fields such as physics or computer-science~~;~~ , but also in the design fields. ~~Lastly~~Ultimately, I believe ~~it~~ out lab may even instill in the average design/art student an appreciation, or even a passion, for the field of mathematics.

 %To fulfil this, we use innovative designs tools and computer visualization, which can show how playful can become fields in mathematics such as algebraic structures, differential geometry and especially topology.

%\section{About the research team}
%The research topics below will be accomplished by diverse students of the institute:design students (first and second degree), computer science students (first and second degrees) and applied mathematics students.
\section{Detailed description of the research}
\subsection{Aim 1: ~~Classifying and defining~~Definition and classification of songs as ~~three-dimensional~~3D objects }
\label{section:music}
In \cite{bridges2022:461,math11204398}, a framework was proposed for mapping a chorus onto a ~~three-dimensional~~3D structure by transforming the guitar choruses of Beatles songs into their respective curves (with constraints). It focused on exploring the total curvature of the chorus curve, which can define the similarity between different choruses. It ~~also~~ can also help the performer ~~to~~ determine the geometric representation they aim to convey through the number of loops and the direction of the curve. In addition, viewing the curve, as shown in Fig.~\ref{fig:Beatles}, offers non-professional audiences a glimpse into the complexity of composing.

In this project, ~~I~~ we intend to produce and formulate the following concepts:
\begin{itemize}
\item ~~Respective~~ Similar to how the curves are obtained in \cite{math11204398~~}.~~}, an~~The~~ oriented polygonal curve ~~is~~ will be obtained by a sequence of vertices. For each ~~two~~ pair of adjacent vertices, a~~n~~ harmonical distance will be defined. ~~By~~ Using the help of an industrial designer, we will define the physical curve ~~with~~ using ~~changing~~ different materials ~~along~~ between the vertices that ~~which~~ will best represent ~~best~~ the harmonical distance. With this approach, we hope ~~we can~~ not only to hear the song but also feel its respective harmonics. This idea ~~can~~ could be especially ~~important~~ meaningful for those who ~~suppfering~~ suffer from hearing loss.
\item ~~As have been done~~We would like to generalize the chorus-based approach in \cite{math11204398} ~~for the chorus, we would like to generalize this idea~~ for the whole song, i.e., we wish to ~~defining~~ define ~~a~~ curves for the chorus, verses, ~~etc~~and so on. In this case, we ~~may get~~could obtain a knot ~~a~~-like curve structure ~~which~~ that we believe can be explored. Lastly, with the help of an industrial designer, we will produce this physical object.
\item ~~Generalize~~ We intend to generalize the idea of curves to surfaces, i.e., each song will be approximated as a surface. From a topological point of view, each of these surfaces is determined by properties as the Euler characteristic number, orientability, etc., and can be produced as a physical object.
\end{itemize}

Commented [K4]: The text switches between "I" and "we" a few times. My suggestion is to use "we" in most places because there will be multiple people involved in the project. If this is the wrong interpretation, it is fine to switch it back.

%
%The classification theorem is another theorem which determine when two surfaces are equivalent from a topological point of view.
%\begin{theorem}
%
%Two compact surfaces are homeomorphic if only if they agree in character of orientability and Euler characteristic number
%and Euler-Poincar\'e characteristic
%
%\end{theorem}
% The book in\cite{classfication} is dedicated to this theorem and students which study the course in algebraic topology. In \cite{conway2008symmetries}, Conway gave another formulation to this theorem.
%\begin{theorem}[The classification theorem for surfaces \cite{conway2008symmetries}]
% \label{conway's clasification theorem}
% Every surface is topologically equivalent to a ``tidey'' one, obtained from a collection of spheres, by adding handles, holes, cross-caps and cross-handels.
%\end{theorem}
%
%In a first glance, these two different formulations of surface classification needs a detailed constructive proof. In \cite{shahar} \footnote{This work has been done with a student under my supervision} we showed how by designing a a modular object we can show even to non professional audience that indeed surfaces can be sorted by the criterion's above; and even that surfaces can be equivalent from a topologically point of view. The user can play with the modular object and obtain different surfaces

which homeomorphic to: sphere klein bottle, M{\"o}bius strip and more, even if the surfaces looks unfamiliar, see Figure (\ref{fig:zip_klein2}).
%\begin{figure}[htp]
%\begin{center}
%\hspace*{-2cm}
%\includegraphics[scale=0.3]{klein_long_zip.jpg}
%\end{center}
%\vskip -0.1in
%\caption{Zipping to a surface homoeomorphic to a Klein bottle. In the first three images a crosscap is zipped, leaving a gap. Since the short zippers now face each other in opposite directions, one of theme must be flipped to form the handle. In the fourth image the `bottom' short zipper in pulled through the gap in the long zipper to meet the `top' short zipper.}
%\label{fig:zip_klein2}
%\end{figure}
%
%This modular object shows to diverse students ideas which for some of them will give the spark to enjoy math and a deeper point of view to advanced mathematics.

\subsubsection{Rationale}

This project will ~~strives~~ strive---with the help of mathematics and industrial ~~design~~ design---to transform music into a tangible physical object. Understanding the structure of music typically requires a great deal of study. In this work, with the help of design tools, we will convert music into objects ~~by relying on their respective chords, which~~ that reflect the complexity in a given song by relying on their respective chords. This 3D visualization can offer non-musicians a ~~glance~~ glimpse into how complicated or simple a piece of music is. We will explore famous songs, especially those in Western music, where the song ~~is~~ generally comprise~~d ofs~~s a verse and chorus. We will show that some songs that sound utterly different can, in fact, be represented by the same object.

Commented [K5]: The original states that chords reflect the complexity of a song. I think you mean to say that the objects reflect the complexity of the song, hence I made this edit. Please check it is the right interpretation. If not, it is fine to reject this change.

\subsubsection{Work plan}

Music can be written as triads (a,b,c). The set of all 24 major and minor triads can be thought of as an abelian group isomorphic to the group $\mathbb{Z}_{12} \times \mathbb{Z}_{2}$. References \cite{mdpi,Hook} give~~s~~ a mathematical formulation for triads.

This research will show that ~~"~~``songs~~"~~'' can be simulated as a collection of curves or surfaces. In this study, the initial input will be songs composed by a sequence of triads (without voicing).

% A triad is constructed by 3 notes each can be represented in $\mathbb{Z}_{12}$. It is well known that we can define a musical group of triad in a similar way to the symmetry group. This group is determined by transposition (which shifting the triad), inversion (respective to modulo 12) and retrograde (which is the triad in reverse order).

A song in Western music is defined (generally) as a chorus and verse, each of which defines a sequence of triads. Each chorus or verse will be considered a closed curve (by defining the location of each triad, which can define a closed curve with a respective total curvature) or a surface with respective topological properties, such as a Gauss curvature ~~and~~ or geodesic curvature (similar~~ly to what was done~~ to the work in \cite{PhysRevLett}).

From the curves point of view, an approximation to the collection of curves will be given. In addition, this sequence of points can be approximated as a surface, and each can be sorted topologically by properties such as Euler characteristic number, orientability, and boundary~~, ~~. F~~f~~or more details about this classification, see \cite{shahar}.

The result will be the sorting of songs by the equivalence relation of curves or surfaces; all representations will be visualized with the help of industrial designers to ~~represnt~~represent best the harmony ~~by ~~using a ~~respective ~~suitable material between adjacent vertices. The team will ~~have~~ need to determine how to exhibit these ideas as an object and ~~to ~~portray to diverse audiences the complexity or simplicity of music.

\subsubsection{Preliminary results}
 \begin{figure}[htp]
\centering
%\vskip -0.8in
  \includegraphics[width=0.5\textwidth]{musical_curves.jpeg}
   %\vskip -0.8in
%Images/
\caption{Our representation of three Beatles songs, which we transformed into $3D$ physical objects~~, ~~. F~~f~~rom left to right: ``Hello G~~g~~oodbye,'' ``All Y~~y~~ou N~~n~~eed I~~i~~s L~~l~~ove,'' and ``Like D~~d~~reamers D~~d~~o.'' $3D$ printing Pla/Sla.}
  \hspace*{2cm}
  \vskip -0.1in
 \label{fig:Beatles}
\end{figure}
 In general, given a triad t_i, where $1 \leq i \leq n$, i.e., the chorus has n triads~~. It~~, this triad can be written as the sequence $t_1,\dots t_n$, i.e., $t_1 \to t_2, \dots ,t_{n-1} \to t_n$, which defines a polygonal curve. In \cite{math11204398}, this chorus polygonal curve has been explored ~~by ~~using ~~the ~~its total curvature~~ point of view~~.

In this proposition, for each chorus ~~which ~~that is defined by a curve, we will define for each ~~two ~~pair of adjacent chords/vertices a harmonic distance, as has been described in \cite{tymoczko2010geometry}. With the help of an industrial designer, each harmonic distance will be represented by a ~~respective ~~suitable material~~, ~~: if the harmonic distance is ``small,'' the edge will be represented by a soft material and pleasant color, and if the harmonic distance is ``large'' then the edge will be represented by a rough or spiky material. The result ~~is ~~will be an object ~~with ~~made of diverse materials, where touching along the object (along the curve) will ~~lead ~~invoke a feeling of ~~to ~~the chorus~~ feel~~. We still need to ~~decide what are~~determine the right materials and metric to use.

Commented [K6]: This is a difficult concept to put into words. Please check the nuance of your meaning is still conveyed. If not, let me know and I will look at it again.

```
% \begin{figure}[htp]
% \begin{center}
% \hspace*{-2cm}
% \includegraphics[scale=0.2]{design_musical_curves_1.jpg}
% \end{center}
% \vskip -0.1in
% \caption{Prototype of a model to explore musical curves.}
% \label{fig:musical_curves}
% \end{figure}

% \begin{figure}[ht!]
% \begin{subfigure}[b]{0.5\linewidth}
```

% \centering
% \includegraphics[width=0.5\linewidth]{musical_curves_1.jpg}
% \caption{}
% \label{curves1}
% \vspace{2ex}
% \end{subfigure}%%
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=0.5\linewidth]{musical_curves_2.jpg}
% \caption{}
% \label{curves2:b}
% \vspace{2ex}
% \end{subfigure}
% \caption{Prototype of the 3d model to explore musical curves.}
% \label{proto}
% \end{figure}
% Another parameter that can be considered is torsion. In this case, we will say that two songs are equal if their respective total curvature (and torsion) are equal. This needs to be done irrespective of it being a verse or a chorus.
%We start by taking a theoretical sequence in $\mathbb{Z}_{12}^{3}$ which starts in a triad (a,b,c) (root) and accumulated a triad which is inversion or retrograde. This sequence can thought of a curve which it starting point and ending point can define the curve, I can find the right presentation such that the staring point and the inversion to become unified, the modeling can be taken to winding number by determine the curve direction, if the direction is positive it $+1$, if its negative its -1. In a similar way an inversion for a monotonically increasing sequence to the inversion triad of the first triad, can be thought of M{\"o}bius strip.
\subsubsection{Pitfalls}

%\textcolor{red}{add the pitfalls of moving from algebraic structure to definition in topology....solution...there a few methods we can help this problem, in previous works i has a few problems similar to that ...which is solved with... }
~~Due to t~~The results in **\cite**{math11204398~~}, ~~} indicate that the aim of defining the feeling of ~~the ~~a chorus along ~~the ~~a physical curve with the help of an industrial designer is reasonable.

Exploring songs as a collection of curves is challenging. In this case, the main question is ~~what is ~~the importance of the knot~~? ~~. ~~Will ~~Moreover, will it give any real insights to musicians or lead to a better understanding ~~to ~~in non-professional audiences?

~~In the case of~~When exploring a chorus as a surface, the ~~resuly~~result will be obtained ~~by ~~using the right triangulation. The result in this case ~~can ~~could ~~be in ~~contradict~~ion to~~ the results in **\cite**{math11204398}.

%connection of the algebraic structure to the curves or surfaces needs to be thoroughly examined. Second, a song can be thought of as a composition of triads, but an average song is composed of a small number of triads. We will have to define an initial data of songs that has enough triads but is still composed of an accessible mathematical point of view.

% To succeed this research, I founded a team which assembled by students from different background: a student in computer science which have a deep understating in music, a student in applied mathematics which needs to understand the algebraic structure and the import a student in industrial design which realize the song to a curve with a proper the right materials.The team need to be committed to the research for at least one year

\subsubsection{Expected outcomes and impact}

This research should yield a new method for generating a tangible visualization (curves or surfaces) of songs. It will present to a wide audience why music can be considerably complex and give an idea of how music relates to mathematical ideas such as algebra, geometry, and topology. It can be transformed from a hearing experience to one of touching an object that reflects the music's internal harmony. It can also provide musicians with a tool for portraying the diverse nature of their music and offer avid audiences who are ~~non-~~not professional musicians~~s but yet experienced audiences~~ a ~~glance~~ glimpse into the complexity of music. ~~Last~~Ultimately, these kinds of tools, ~~with~~ using our technique and the right materials, may one day allow deaf people to enjoy a song, not by hearing it but by feeling it.
% \subsubsection{Time table}
% This project is just started. I am already
\subsection{Aim 2: Gradient topology}
\label{section_Gradient_topology}
In art and design, a gradient is a smooth transition from one color to another. It ~~gives~~ enables ~~the~~ an artist/designer to add a soft feel and uniqueness to their object. It also~~,~~ ~~has~~ leads to eye-catching and memorable visual designs, ~~while~~ whereas solid colors can be thought of as stiff colors. ~~It~~ Gradients can be applied in cases where the artist ~~trying~~ wishes to transmit shade or light on a given product, create a focal point, ~~etc.~~or create some other type of effect.~~,~~ F~~f~~or more details about surface classification, see \cite{sherin2012design,Topology}.

It has led us ~~to think~~ (mathematicians and designers~~), ~~) to wonder if can we formulate different gradients for color ~~by~~ using a mathematical rule? Even though ~~usually~~ gradient is usually related to geometry, our approach ~~leaned~~ rests on fundamental polygons in topology, which represent different classes of surface~~s classification~~s (toruses, Klein bottles, ~~etc.~~and so on). Many ~~works~~ studies have ~~been done about~~considered ~~Suduku~~Sudoku, which can be related to a visual gradient solution for a given matrix (see \cite{davis2006mathematics,suduko2,suduko3}). We believe that many ideas and solutions can help us and vice versa~~, see \cite{davis2006mathematics,suduko2,suduko3}~~. We further believe that this research will ~~shows~~ demonstrate the importance of involving other fields that mathematicians are not familiar with~~,~~ (in this case design), which will inspire~~s~~ the ~~formulating~~ formulation of new mathematical explorations.

% \begin{figure}[htp]
% \begin{center}
% %\hspace*{-2cm}
% \includegraphics[scale=0.2]{rothko2.jpg}
% \end{center}
% \vskip -0.1in
% \caption{A Rothko art painting}
% \label{fig:roth}
% \end{figure}

Commented [K7]: The term “avid” is not quite the same as “experienced,” but I think it expresses your meaning.

Commented [K8]: I think these are specific to Soduko, so I moved them up. Please check this is correct.

```
%In this manuscript, we introduce different multi-coloring symmetrical
patterns that are obtained only by a transformation on an initial location
matrix. These different patterns are obtained by different algebraic
structures, which lead to this symmetries results. First, we generalize a
result of \cite{Ian} to RGB and CMYK color formats by the properties of the
Quaternion group. Second, we use finite rings to show how it produces a
perception of weaving symmetry patterns. Lastly, we use the Quaternion group
to produce spherical patterns in the RGB format.
```

```
% \begin{figure}[ht!]
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=0.75\linewidth]{mod_38.jpg}
% \caption{}
% \label{fig7:a}
% \vspace{4ex}
% \end{subfigure}%%
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=0.75\linewidth]{mod_17.jpg}
% \caption{}
% \label{fig7:b}
% \vspace{4ex}
% \end{subfigure}
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=0.75\linewidth]{mod_23.jpg}
% \caption{}
% \label{fig7:c}
% \end{subfigure}%%
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=0.75\linewidth]{mod_25.jpg}
% \caption{}
% \label{fig7:d}
% \end{subfigure}
% \caption{In \cite{patterns}, given a location matrix, the finite ring with
a respective transformation defines a color in gray scale to each location,
which leads to these different patterns under different constraints.}
% \label{fig7}
% \end{figure}
```

\subsubsection{Rationale}
 We will define for the first time a language of visual gradients, which is influenced by design ideas combined with topology and combinatorics. We will show how to construct different types of visual gradients given a fundamental polygon for different initial states, such as M\"{o}bius strips or~~,~~ Klein bottle~~s, etc.~~. Further, given an initial state of a gradient (~~which will be~~as defined in Section~\ref{sec:gradient_initial}), we will show which topological gradient~~s~~ constructions can be obtained.

Commented [K9]: Please check. In some places, you use the term “design gradient,” and in others, you use the term “visual gradient.” If they are different things, that is fine. If they are the same term, it would be best to choose one term and stick with it.

Once we have defined a good language for gradients, we will be able to
define harder questions and obtain deeper results, and our hope is that our
point of view will open the door for systematic research in this area, both
~~from~~ in mathematics and design.

Lastly, topology has ~~a lot~~much to offer ~~to~~ art and design, as demonstrated in
\cite{sequin1,sequin2~~}, ~~}; we will show how it can be a ~~real~~ practical tool
~~to~~ for the average designer.
\subsubsection{Work plan}
We first need to define a language that connects ~~between~~ visual gradients and
~~a~~ fundamental polygons. We also need to define a good filter for~~ation for~~ the
space of possible initial states, which is identified ~~with~~ using a partial
defined matrix. In the next step, we will ~~try to~~ classify which topological
visual gradients could be obtained from each step in the ~~filtration~~filtering.
Lastly, the industrial designer in the team will apply our formulation for a
given product~~ which~~ that is homeomorphic to a given surface.

\subsubsection{Preliminary results}
\label{sec:gradient_initial}
~~These are~~Here, we present the basic definitions and initial results.
\begin{definition}
Given a cell (i,j) in a matrix $n \times n$. ~~We~~ we define ~~ cell'~~its~~s~~
neighbors as all the cells it borders with horizontally, vertically, or
diagonally as $pixel(i,j)$, $neighbor(pixel(i,j))$.
\end{definition}
Notice that by this definition, an interior pixel ~~have~~ has eight neighbors,
~~see~~ as shown in Fig.~\ref{fig:neighboors_gradient}.
\begin{figure}[htp]
\begin{center}
%\hspace*{-2cm}
\includegraphics[scale=0.2]{Figures/Neigboors.png}
\end{center}
\vskip -0.1in
\caption{Cell 1's ~~neigboors~~neighbors are indicated in yellow, cell 2's
~~neigboors~~neighbors ~~in~~ are blue, and the green cell is a common
~~neigboor~~neighbor to both.}
\label{fig:neighboors_gradient}
\end{figure}
In gray-scale, a continuous color scale ~~will~~ is ~~be~~ defined that starts at
zero and ~~increasing~~ increases a~~i~~s a ~~constanst~~constant natural number C~~,~~
such that $C\cdot n \leq 255$; for~~.~~ e~~E~~xample~~s~~, $0,1,2, \dots 255$ or~~,~~
$0,5,10 \dots255$.
\begin{definition} [Visual Gradient]
Given a pixel (i,j) and a continuous color scale~~. ~~, ~~If~~ for each pixel
neighbor $$\|{pixel(i,j)-neighbor(pixel(i,j))\|}= 0 \; \text{or} \;C$$
\end{definition}
To explore this connection, we define the following~~ cases~~:
\begin{definition}[Initial state ~~term~~ for gradient~~s~~ topology]
An initial state of a gradient is a partial~~ly~~ field matrix.
\end{definition}
\begin{definition}
A matrix $n\times n$ is called a full initial state if all the borders of the
matrix are full. It is a ~~Partial~~ partial initiate state if ~~it~~ the borders are
only partially given.
\end{definition}
For example, see Fig.~\ref{more_cylinder_grad}.
\begin{figure}[h!bp]

Commented [K10]: I recommend this alternative because the definitions are for several things (states and gradients), not a well-defined set of cases.

Commented [K11]: Please check. Should this also be "initial state" to match the term used in the definition? If not, please ignore this comment.

Commented [K12]: I also think you might mean "An initial state for a gradient is a partial field matrix." If this is the wrong interpretation, please ignore this comment.

Commented [K13]: Please check. I think this should refer to Fig. 4, but it refers to Fig. 5 instead.

```latex
\begin{minipage}[b]{0.3\textwidth}
      \includegraphics[width=\textwidth]{Figures/full.png}
            \subcaption{Full initial state. ~~In t~~This ~~case ~~state ~~it may~~could
lead to a Klein ~~buttle~~bottle.} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:full}
\end{minipage}
~
\qquad
%\quad %add desired spacing between images, e. g., ~, \quad, \qquad, \hfill
etc.
\begin{minipage}[b]{0.3\textwidth}
      \includegraphics[width=\textwidth]{Figures/partial.png}
            \subcaption{Partial initial state. ~~It ~~This state ~~may ~~could lead
to different topological surfaces.} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:partial}
\end{minipage}
\qquad

\caption{~~This ~~The initial states of a gradient can define the topological
surface.}
\label{more_cylinder_grad}
\end{figure}
\begin{definition}
An initial state will be called monotone if the initial values in each row or
column~~s~~ define~~s~~ a strictly monotonic sequence.
\end{definition}

\begin{definition}
Let $X$ be a topological surface. An initial state $A$ will be called an
initial state of $X$ if the edges of $A$ define~~s~~ the  fundamental rectangle
of $X$.
\end{definition}

% \begin{figure}[h!tbp]
%     \centering
%     \fbox{\includegraphics[width=2.4in]{monotone.png}}
%     \caption{A monotone vector}
%     \label{fig:monotone}
% \end{figure}


The following definition ~~will ~~connects the visual gradient and topology.
\begin{definition}[~~An i~~Initial ~~topology ~~topological gradient]
Given a topologocal~~y~~ surface $X$ with a respective fundamental polygon
gradient~~. ~~, a~~A~~n initial $X$ state for a gradient matrix $n$~~,~~ is an initial
state aligned with the fundamental polygon.
\end{definition}
~~Now w~~We are now ready to define visual gradients as respective topological
surfaces. We ~~will ~~start with the most intuitive one, a cylinder.
\begin{definition}
A cylinder gradient is a gradient ~~which ~~that is defined by the fundamental
polygon with only two parallel edges in the same direction, i.e., it
represents a cylinder $\forall 1\leq j \leq n: pixel(1,j)=pixel(n,j)$. A
rotation of this gradient is ~~from ~~of the same type.
```

```
\end{definition}
\begin{figure}[h!bp]
\centering
\begin{minipage}[b]{0.29\textwidth}
      \includegraphics[width=\textwidth]{Figures/cylinder_1.jpg}
            \subcaption{Cylinder} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:cylinder_1}
\end{minipage}
~
\qquad
%\quad %add desired spacing between images, e. g., ~, \quad, \qquad, \hfill
etc.
\begin{minipage}[b]{0.29\textwidth}
      \includegraphics[width=\textwidth]{Figures/cylinder_2.jpg}
            \subcaption{Cylinder} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:cylinder_2}
\end{minipage}
\qquad
\begin{minipage}[b]{0.29\textwidth}
      \includegraphics[width=\textwidth]{Figures/RP_1.jpg}
            \subcaption{sphere} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:sphere}
\end{minipage}

% \begin{minipage}[b]{0.3\textwidth}
%     \includegraphics[width=\textwidth]{Images/big model stage 3.jpg}
%           \subcaption{A mechanism closeup} % Add subcaption text if
desired, or use \subcaption* to suppress (a), (b), etc. labels
%           \label{fig:2d}
% \end{minipage}

\caption{Examples of visual gradients for which the initial state defines a
topological surface.}
\label{more_cylinder_grad}
\end{figure}
```

In Fig~~ures~~s.~~ ~~~\ref{fig:cylinder_1} and \ref{fig:cylinder_2}, we can see that the cylinder gradient is not unique, and this made us wonder ~~`Can~~whether we can ~~give~~ determine an upper and lower bound to the number of gradients that can be obtained for a given topological surface~~?'~~.

```
%  \begin{figure}[htp]
% \centering
% %\vskip -0.8in
%   \includegraphics[width=0.4\textwidth]{Figures/RP_1.jpg}
%    %\vskip -0.8in
% %Images/
% \caption{A gradient which is defined in the gray color space. In this case
can be considered as the oriented square of a real projective plane.}
%   \hspace*{2cm}
%   \vskip -0.1in
%  \label{fig:RP}
% \end{figure}
```

We ~~will ~~here give a ~~glance ~~glimpse ~~to ~~of the power~~ful~~ of the language we are trying to formulate.
\begin{proposition}
Given a monotonic initial state for a given image $n \times n$ and $n \leq 256$, ~~then ~~X must be a sphere.
\end{proposition}
We ~~decide to ~~omit the proof~~, ~~; for an example ~~for ~~of a sphere gradient, see Fig.~\ref{fig:sphere}.

```
% \begin{figure}[ht!]
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=1\linewidth]{side-by-side-1.png}
% \caption{}
% \label{sbs1}
% \vspace{4ex}
% \end{subfigure}%%
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=1\linewidth]{side-by-side-2.png}
% \caption{}
% \label{sbs2}
% \vspace{4ex}
% \end{subfigure}
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=1\linewidth]{side-by-side-3.png}
% \caption{}
% \label{sbs3}
% \end{subfigure}%%
% \begin{subfigure}[b]{0.5\linewidth}
% \centering
% \includegraphics[width=1\linewidth]{side-by-side-5.png}
% \caption{}
% \label{sbs4}
% \end{subfigure}
% \caption{Testing Rothko's art works and a given color transformation. In the left side of every sub figure the original art work, in the right side the transformed one.}
% \label{side_by_side}
% \end{figure}
```

\subsubsection{Pitfalls}
This research involves different disciplines, each of which requires a ~~specialty~~specialism. We may add ~~to the team ~~more researchers from mathematics, computer science, and design to the team to ~~make ~~ensure that progress is obtained.
\subsubsection{Expected outcomes and impact}
The expected outcome is a systematic math treatment for the visual gradient. This proposal has many different research outcomes. From the design point of view, understanding the mathematics can help the designer ~~well ~~customize which visual gradient is best to apply in a given product. From the mathematical point of view, our formulation connects fields which ~~supposed to be so~~are seen as very different from one ~~to ~~another~~: ~~, e.g., geometry, topology, and combinatorics. In addition, there are hundreds of publications

exploring ~~Suduko~~Sudoku, and we believe that the scientific community in this field will ~~find a lot of~~be strongly interested in our language.
Equally important, it will encourage mathematicians to open their mind and learn about diverse fields (in our case, the fields of design~~/~~ and art) that might not seem related to their main research. It may also be considered in the curriculum for undergraduate students in mathematics or design~~, trying to~~ to ~~show~~ demonstrate how intermediate mathematics can be applied with the right guidelines.

\subsection{Aim 3: Dynamical Til~~l~~ing}
\label{section_dynamical_tilying}
Usually, industrial design innovation is related to materials and ~~AI~~artificial intelligence. When designers deal with mathematics, it usually relates to the geometric properties of the product, ~~while~~ even though mathematics has a variety of tools that can be used~~;~~, such as algebra, topology, etc.
This research shows that innovation can come from unexpected places, such as group theory, which can define a dynamical tiling ~~between~~ with different stages. This topic is essential ~~to~~ if designers are to increase their toolbox not only in the ~~finish~~ final steps of defining a pattern but also in the initial steps of planning a product ~~which~~ that ~~have~~ has tiling~~s~~ properties.

We intend to define a geodesic dome ~~which~~ that is obtained by a tiling~~-~~ group, which can define a dynamical movement~~-~~ based on the respective sub-group. We may consider defining a proper mechanism that determines the movement, as has been done in \cite{bridges2023:413}. ~~As~~ Defined using mathematics, our dynamical tiling is not designed for a specific task. Instead, it is a concept that can be applied to the specific demand of a designer who intends to innovate a product with certain properties, such as reduction~~ and~~ /expansion or~~,~~ ~~exposure~~ exposing/~~or hiding~~concealing~~, and more~~.

```
\begin{figure}[h!bp]
\centering
\begin{minipage}[b]{0.3\textwidth}
	\includegraphics[width=\textwidth]{Figures/big model stage 1.jpg}
		\subcaption{Initial state, with signature $*632$} % Add 
subcaption text if desired, or use \subcaption* to suppress (a), (b), etc. 
labels
		\label{fig:big_model_1}
\end{minipage}
~
\quad %add desired spacing between images, e. g., ~, \quad, \qquad, \hfill 
etc.
\begin{minipage}[b]{0.3\textwidth}
	\includegraphics[width=\textwidth]{Figures/big model stage 2.jpg}
		\subcaption{Middle state, which leads to signature $632$} % Add 
subcaption text if desired, or use \subcaption* to suppress (a), (b), etc. 
labels
		\label{fig:big_model_2}
\end{minipage}
\quad
\begin{minipage}[b]{0.3\textwidth}
	\includegraphics[width=\textwidth]{Figures/big model stage 3.jpg}
```

```
            \subcaption{O~~The model o~~pen ~~stage~~state, with signature $*632$} %
Add subcaption text if desired, or use \subcaption* to suppress (a), (b),
etc. labels
            \label{fig:big_model_3}
\end{minipage}
% \begin{minipage}[b]{0.3\textwidth}
%     \includegraphics[width=\textwidth]{Images/big model stage 3.jpg}
%           \subcaption{A mechanism closeup} % Add subcaption text if
desired, or use \subcaption* to suppress (a), (b), etc. labels
%           \label{fig:2d}
% \end{minipage}

\caption{Our planar model defines, ~~with ~~using a proper mechanism, a dynamical
movement between tilings (made from PLA). When ~~it ~~the mechanism reaches the
end of the rail, it ~~gives ~~yields an extended pattern of signature $*632$.}
\label{big_model}
\end{figure}

% \begin{figure}[htp]
% \center
% \input{Min_sum_hex_sq.tex}
% \caption{A Minkowski sum of a hexagon and a rectangle}
% \label{img:rec_min_sum}
% \label{fig:req}
% \end{figure}


\subsubsection{Work plan}
 In \cite{conway2008symmetries}, the signatures of patterns~~/~~ and tilings and
their respective group~~s~~ relationships have been defined. In this project, we
intend to apply some of these relationships and define a dynamical movement
between different stages, not just ~~in ~~on ~~the ~~a plane~~, but especially ~~ but
also on spherical surfaces, especially ~~for ~~geodesic domes. The mathematical
exploration is based on the relations of spherical patterns (which are
related to domes), i.e., the respective groups and sub-groups. To obtain the
geodesic dome, we first consider a ~~different ~~deforming plane model, as shown
in Fig.~\ref{fig:dome_1}, which we believe can give us the knowledge of how
to define the respective mechanism.
```

Commented [K14]: Please also check. I think this should be Fig. 6, not Fig. 7.

```
\subsubsection{Preliminary results}
In \cite{bridges2023:413}, we discussed the relations ~~in ~~among  planar
patterns, and show how a dynamical movement can be obtained between the
hexagonal regular lattice which is defined by reflections, rotation, and
translation (signature $*632$), and its sub-group, which is defined by
rotation and translations (signature $632$), as demonstrated in
\cite{Fletcher}.

Before moving to spherical patterns, we first delve into planar patterns,
since ~~it ~~a planar pattern can be considered ~~as ~~to be a local approximation of
~~the ~~a spherical pattern. We need to decide which planar pattern and relation
is best ~~to ~~deform~~ed~~, not only ~~by ~~according to mathematics but also ~~in~~
according to ~~the ~~material design.

In Fig.~\ref{fig:dome_1}, we are given a local representation of a spherical
pattern with the respective signature~~, ~~. ~~by ~~Using this basic model, we ~~trying~~
```

```latex
attempt to evaluate what is the ``optimal'' movement with the help of the
respective groups.

In Fig.~\ref{khushbu}, planar patterns ~~have been considered by~~implemented
with various materials are considered. In Fig.~\ref{fig:khus_1}, we consider
paper and brass fasteners, which ~~is ~~are flexible enough~~, such that~~ to obtain
a deformation~~ can be obtained~~. We ~~hope ~~believe that ~~it ~~this approach will
give us another ~~direction ~~avenue in spherical pattern approximation to
explore~~to spherical pattern approximation~~. In Fig.~\ref{fig:khus_2}, we use
laser-cut wood and brass fasteners; ~~which ~~this leads to a ~~-~~scissor~~-~~-linkage
mechanism for a pentagon ~~and ~~that opens and closes. Since~~,~~ in some ~~of the~~
spherical patterns, pentagons play a major role, w~~W~~e believe that this kind
of experiment~~s~~ can lead to the desired mechanism. In Fig.~\ref{fig:khus_4},
we combine~~ing~~ the previous steps~~, we ~~ and ~~try ~~attempt to approximate
signature $*532$.




\begin{figure}[htp]
\begin{center}
\hspace*{-2cm}
\includegraphics[scale=0.15]{Figures/geodesic.jpeg}
\end{center}
\vskip -0.1in
\caption{Our first naive experiment in a spherical pattern~~s, ~~ with signature
$*532$. By ~~playing ~~exploring the object, e.g., ~~as dismissing~~removing the
triangles, we ~~trying ~~attempt to predict the mechanism.}
\label{fig:dome_1}
\end{figure}

\begin{figure}[h!bp]
%\centering
\hspace*{-3cm}
%\fbox{
\begin{minipage}[b]{0.24\textwidth}
      \includegraphics[width=\textwidth]{Figures/origami1.jpeg}
            \subcaption{} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:khus_1}
\end{minipage}
~
\quad %add desired spacing between images, e. g., ~, \quad, \qquad, \hfill
etc.
\begin{minipage}[b]{0.27\textwidth}
      \includegraphics[width=\textwidth]{Figures/mechanism1.jpeg}
            \subcaption{} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:khus_2}
\end{minipage}
\quad
\begin{minipage}[b]{0.3\textwidth}
      \includegraphics[width=\textwidth]{Figures/mechanism2.jpeg}
            \subcaption{} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:khus_3}
```

```latex
\end{minipage}
\qquad
\begin{minipage}[b]{0.24\textwidth}
      \includegraphics[width=\textwidth]{Figures/hex_approx_1.jpeg}
            \subcaption{} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
            \label{fig:khus_4}
\end{minipage}
\qquad
% \begin{minipage}[b]{0.3\textwidth}
%     \includegraphics[width=\textwidth]{Figures/combiation1.jpeg}
%           \subcaption{} % Add subcaption text if desired, or use
\subcaption* to suppress (a), (b), etc. labels
%           \label{fig:big_model_3}
% \end{minipage}
% \begin{minipage}[b]{0.3\textwidth}
%     \includegraphics[width=\textwidth]{Images/big model stage 3.jpg}
%           \subcaption{A mechanism closeup} % Add subcaption text if
desired, or use \subcaption* to suppress (a), (b), etc. labels
%           \label{fig:2d}
% \end{minipage}
%}
\caption{Exploring movement, deformation, and mechanism.}
\label{khushbu}
\end{figure}



% \begin{figure}[htp]
% \begin{center}
% \hspace*{-2cm}
% \includegraphics[scale=0.7]{table.jpeg}
% \end{center}
% \vskip -0.1in
% \caption{Initial result. This table base was constructed by an arbitrary
closed curve and a chosen Minkowski sum such that a stable table is formed.}
% \label{fig:table}
% \end{figure}
```

\subsubsection{Pitfalls}
~~If we would like~~To develop a ~~the~~ dynamical geodesic dome ~~to be applied to~~for real ~~uses~~ applications, the construction needs to be efficient. ~~As an~~For example, if the mechanism is inside the dome, the ~~inner side of~~space inside the dome will be ~~lack of space~~reduced.

\subsubsection{Expected outcomes and impact}
This research will ~~gives~~yield new mathematical tools ~~to~~ for the average industrial designer. The designer will be able to choo~~o~~se~~s~~ a given pattern in which a movement is required in a physical object ~~to~~ for a given purpose. ~~By~~ Using our mathematical language, the pattern can be transformed into the respective fundamental area, til~~y~~ing, and signature. Each such pattern ~~of which (pattern),~~ has a sub-group that will be ~~the~~ a natural candidate for the dynamical movement ~~which is~~ required.

\section{Summary}

In this project, I seek to find a symbiotic relationship between design and
mathematics~~ in this project~~.
In Section~\ref{section:music}, I describe how industrial design ~~by ~~based on
mathematical foundations helps to give a tangible feel and ~~visual ~~look to
 ~~to ~~music.
Section~\ref{section_Gradient_topology} ~~shows ~~presents how design-based
definitions can help ~~to ~~formulate new ideas in mathematics.
Finally, Section~\ref{section_dynamical_tilying} ~~shows ~~discusses how abstract
~~algebriac~~algebraic ideas can be adapted to industrial design.

% music, design (especially industrial) can be merged with math as a means to
show diverse audiencees the relevance of mathematics to their field and that
it indeed can be tangible and applied in exciting ways in their field, with
the help of modern design and computer software tools.

\bibliographystyle{plain}
%\setlength{bibsep}{1.5pt}
%\begin{spacing}{1.5}
\bibliography{ISF_bib}
%\end{spacing}

\end{document}