1. Scientific background

Mathematics is key to many fields and is relevant to the vast majority of tertiary students. Yet most students shy away from this discipline, viewing it as a field that merely deals with quantities. Beyond being a factor in student undergraduate course choice, this bias also limits the ability of students to realize the full potential of the fields in which they have chosen to major. In many faculties, students are not aware of what mathematics has to offer—that it indeed deals with quantities but also with patterns, structures, changes and space.

Perhaps the greatest lack of undergraduate math concepts can be found in the faculties of art and design, which usually do not include mathematical ideas in their curriculum (other than basic geometry). Designers, especially industrial designers, are educated in academia to innovate new products and features. They are driven by this objective to push their boundaries with the help of other scientific domains, including materials engineering (especially mockups in 3D printing), artificial intelligence, mechanics, and other fields. From my point of view, mathematics has a variety of tools (algebra, topology, etc.) that are just waiting for the right open-minded designer to apply them. Theoretical mathematical tools can be considered not only for patterns in the finishing process of a given product but also in the initial steps of planning a product. In some cases, questions such as "Is it possible to define a product X with properties Y?" can be answered in the planning process using mathematical justifications, as described in [3].

This proposal will focus on both the symbiotic relationship between mathematics and design and how the tools of each of these distinct fields can lead to scientific innovations in the other.

My vision is to propose and establish a field similar to mathematical physics and mathematical biology called *mathematical design* and show how important it is that intermediate math is a part of the curriculum in different courses on art and design.

2. Objectives and significance

The main objective of this project is to connect mathematics, art, and design. We will show how mathematics with computational tools can define innovation in design and art, and more surprisingly, how design concepts can inspire the development of new mathematical ideas. In this proposal, the focus is on the following three topics:

(1) **Aim 1: Definition and classification of songs as three-dimensional (3D) objects**. Can a given song in Western music be modeled as a collection of curves or surfaces, or even be defined as a tangible object? If they indeed can be modeled in a pure mathematical fashion, can we sort songs using equivalence relations? This research involves music, industrial design, differential geometry, algebra, and topology.

(2) **Aim 2: Gradient topology**. A gradient is an important concept in mathematics, and surprisingly, this concept is also well defined from a designer's point of view as a gradual change in color (which contains the mathematical definition) in a given image. Although this simple design gradient, as shown

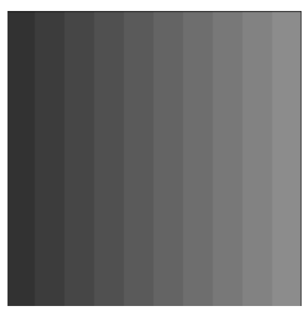


Figure 1. A classical visual gradient in a grayscale color space. This case can be considered to be the fundamental polygon of a cylinder, since the upper and lower sides are in the same color direction.

in Fig. 1, has a geometrical property, it reminds us as mathematicians of the construction of a cylinder from the respective fundamental polygon, which is obtained by attaching the edges with identical colors. This observation led us to wonder if it is possible to define design-gradients for different topological surfaces (toruses, Klein bottles, etc.), and if the answer is yes, whether we assign an upper and lower bound to the number of gradients that exist for each topological surface. This research involves design, topology, combinatorics, and complexity; all are influenced by design concepts.

(3) **Aim 3: Defining dynamical tiling in industrial design**. It has been shown that algebraic structures can help designers in the planning stages to determine if a dynamical transformation of the components can be obtained simply by using a respective mechanism that defines movement between different patterns/arrangements, each of which accomplishes a different goal, as introduced in [1, 3]. This approach can be applied to folding tables (reduction and expansions), lightning systems (exposure and concealment), and other design tasks. We intend to generalize this result not only for planar patterns but also for spherical patterns and especially geodesic domes. This research involves industrial design (3D visualization and mechanism construction), differential geometry, and groups.

3. About the lab

The Lab for Designing Mathematics, which I head, is a multidisciplinary research lab focused on ideas that involve advanced mathematics and design (especially industrial design). My team uses diverse tools from various faculties to achieve this aim, from computer science to industrial design. To advance our goal, we collaborate with various departments on campus, such as design (first and second degrees), computer science (first and second degrees), and applied mathematics, as well as other institutes. The team aims to connect research fields that are traditionally perceived as starkly different, e.g., math and design, including music and art design. We are driven by the belief that our efforts can aid in the dissemination of intermediate mathematics concepts among designers and artists and, of course, help apply non-trivial mathematical ideas not only in traditionally connected fields such as physics or computer-science, but also in the design fields. Ultimately, I believe out lab may even instill in the average design/art student an appreciation, or even a passion, for the field of mathematics.

4. Detailed description of the research

4.1. **Aim 1: Definition and classification of songs as 3D objects.** In [5, 8], a framework was proposed for mapping a chorus onto a 3D structure by transforming the guitar choruses of Beatles songs into their respective curves (with constraints). It focused on exploring the total curvature of the chorus curve, which can define the similarity between different choruses. It can also help the performer determine the geometric representation they aim to convey through the number of loops and the direction of the curve. In addition, viewing the curve, as shown in Fig. 2, offers non-professional audiences a glimpse into the complexity of composing.

In this project, we intend to produce and formulate the following concepts:

- Similar to how the curves are obtained in [8], an oriented polygonal curve will be obtained by a sequence of vertices. For each pair of adjacent vertices, a harmonical distance will be defined. Using the help of an industrial designer, we will define the physical curve using different materials between the vertices that will best represent the harmonical distance. With this approach, we hope not only to hear the song but also feel its respective harmonics. This idea could be especially meaningful for those who suffer from hearing loss.
- We would like to generalize the chorus-based approach in [8] for the whole song, i.e., we wish to define curves for the chorus, verses, and so on. In this case, we could obtain a knot-like curve structure that we believe can be explored. Lastly, with the help of an industrial designer, we will produce this physical object.
- We intend to generalize the idea of curves to surfaces, i.e., each song will be approximated as a surface. From a topological point of view, each of these surfaces is determined by properties as the Euler characteristic number, orientability, etc., and can be produced as a physical object.

4.1.1. *Rationale.* This project will strive—with the help of mathematics and industrial design—to transform music into a tangible physical object. Understanding the structure of music typically requires a great deal of study. In this work, with the help of design tools, we will convert music into objects that reflect the complexity in a given song by relying on their respective chords. This 3D visualization can offer non-musicians a glimpse into how complicated or simple a piece of music is. We will explore famous songs, especially those in Western music, where the song generally comprises a verse and chorus. We will show that some songs that sound utterly different can, in fact, be represented by the same object.

4.1.2. *Work plan.* Music can be written as triads (a, b, c). The set of all 24 major and minor triads can be thought of as an abelian group isomorphic to the group $\mathbb{Z}_{12} \times \mathbb{Z}_2$. References [2, 9] give a mathematical formulation for triads.

This research will show that "songs" can be simulated as a collection of curves or surfaces. In this study, the initial input will be songs composed by a sequence of triads (without voicing).

A song in Western music is defined (generally) as a chorus and verse, each of which defines a sequence of triads. Each chorus or verse will be considered a closed curve (by defining the location of each triad, which can define a closed curve with a respective total curvature) or a surface with respective topological properties, such as a Gauss curvature or geodesic curvature (similar to the work in [4]).

From the curves point of view, an approximation to the collection of curves will be given. In addition, this sequence of points can be approximated as a surface, and each can be sorted topologically by properties such as Euler characteristic number, orientability, and boundary. For more details about this classification, see [11].

The result will be the sorting of songs by the equivalence relation of curves or surfaces; all representations will be visualized with the help of industrial designers to represent best the harmony using a suitable material between adjacent vertices. The team will need to determine how to exhibit these ideas as an object and portray to diverse audiences the complexity or simplicity of music.

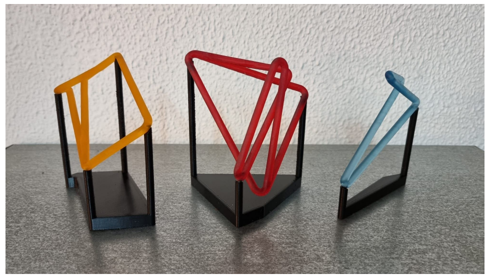


FIGURE 2. Our representation of three Beatles songs, which we transformed into $3D$ physical objects. From left to right: "Hello Goodbye," "All You Need Is Love," and "Like Dreamers Do." $3D$ printing Pla/Sla.

4.1.3. *Preliminary results.* In general, given a triad t_i, where $1 \leq i \leq n$, i.e., the chorus has n triads, this triad can be written as the sequence $t_1, \ldots t_n$, i.e., $t_1 \to t_2, \ldots, t_{n-1} \to t_n$, which defines a polygonal curve. In [8], this chorus polygonal curve has been explored using its total curvature.

In this proposition, for each chorus that is defined by a curve, we will define for each pair of adjacent chords/vertices a harmonic distance, as has been described in [16]. With the help of an industrial designer, each harmonic distance will be represented by a suitable material: if the harmonic distance is "small," the edge will be represented by a soft material and pleasant color, and if the harmonic distance is "large" then the edge will be represented by a rough or spiky material. The result will be an object made of diverse materials, where touching along the object (along the curve) will invoke a feeling of the chorus. We still need to determine the right materials and metric to use.

4.1.4. *Pitfalls.* The results in [8] indicate that the aim of defining the feeling of a chorus along a physical curve with the help of an industrial designer is reasonable.

Exploring songs as a collection of curves is challenging. In this case, the main question is the importance of the knot. Moreover, will it give any real insights to musicians or lead to a better understanding in non-professional audiences?

When exploring a chorus as a surface, the result will be obtained using the right triangulation. The result in this case could contradict the results in [8].

4.1.5. *Expected outcomes and impact.* This research should yield a new method for generating a tangible visualization (curves or surfaces) of songs. It will present to a wide audience why music can be considerably complex and give an idea of how music relates to mathematical ideas such as algebra, geometry, and topology. It can be transformed from a hearing experience to one of touching an object that reflects the music's internal harmony. It can also provide musicians with a tool for portraying the diverse nature of their music and offer avid audiences who are not professional musicians a glimpse into the complexity of music. Ultimately, these kinds of tools, using our technique and the right materials, may one day allow deaf people to enjoy a song, not by hearing it but by feeling it.

4.2. **Aim 2: Gradient topology.** In art and design, a gradient is a smooth transition from one color to another. It enables an artist/designer to add a soft feel and uniqueness to their object. It also leads to eye-catching and memorable visual designs, whereas solid colors can be thought of as stiff colors. Gradients can be applied in cases where the artist wishes to transmit shade or light on a given product, create a focal point, or create some other type of effect. For more details about surface classification, see [10, 13].

It has led us (mathematicians and designers) to wonder if can we formulate different gradients for color using a mathematical rule? Even though gradient is usually related to geometry, our approach rests on fundamental polygons in topology, which represent different classes of surfaces (toruses, Klein bottles, and so on). Many studies have considered Sudoku, which can be related to a visual gradient solution for a given matrix (see [7, 12, 17]). We believe that many ideas and solutions can help us and vice versa. We further believe that this research will demonstrate the importance of involving other fields that mathematicians are not familiar with (in this case design), which will inspire the formulation of new mathematical explorations.

4.2.1. *Rationale.* We will define for the first time a language of visual gradients, which is influenced by design ideas combined with topology and combinatorics. We will show how to construct different types of visual gradients given a fundamental polygon for different initial states, such as Möbius strips or Klein bottles. Further, given an initial state of a gradient (as defined in Section 4.2.3), we will show which topological gradient constructions can be obtained.

Once we have defined a good language for gradients, we will be able to define harder questions and obtain deeper results, and our hope is that our point of view will open the door for systematic research in this area, both in mathematics and design.

Lastly, topology has much to offer art and design, as demonstrated in [14, 15]; we will show how it can be a practical tool for the average designer.

4.2.2. *Work plan.* We first need to define a language that connects visual gradients and fundamental polygons. We also need to define a good filter for the space of possible initial states, which is identified using a partial defined matrix. In the next step, we will classify which topological visual gradients could be obtained from each step in the filtering. Lastly, the industrial designer in the team will apply our formulation for a given product that is homeomorphic to a given surface.

4.2.3. *Preliminary results.* Here, we present the basic definitions and initial results.

Definition 4.1. Given a cell (i, j) in a matrix $n \times n$. we define its neighbors as all the cells it borders with horizontally, vertically, or diagonally as $pixel(i, j)$, $neighbor(pixel(i, j))$.

Notice that by this definition, an interior pixel has eight neighbors, as shown in Fig. 3. In grayscale, a continuous color scale is defined that starts at zero and

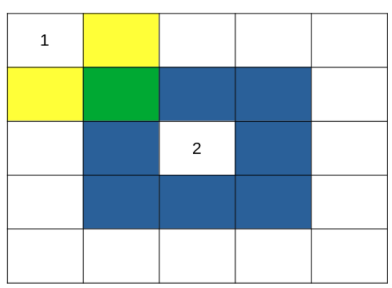


FIGURE 3. Cell 1's neighbors are indicated in yellow, cell 2's neighbors are blue, and the green cell is a common neighbor to both.

increases as a constant natural number C such that $C \cdot n \leq 255$; for example, $0, 1, 2, \ldots 255$ or $0, 5, 10 \ldots 255$.

Definition 4.2 (Visual Gradient)**.** Given a pixel (i, j) and a continuous color scale, for each pixel neighbor

$$\|pixel(i,j) - neighbor(pixel(i,j))\| = 0 \text{ or } C$$

To explore this connection, we define the following:

Definition 4.3 (Initial state for gradient topology)**.** An initial state of a gradient is a partial field matrix.

Definition 4.4. A matrix $n \times n$ is called a full initial state if all the borders of the matrix are full. It is a partial initiate state if the borders are only partially given.

For example, see Fig. 5.

1	2	3	2	3
2				2
3				1
2				2
3	2	1	2	1

(A) Full initial state. This state could lead to a Klein bottle.

1	2	3	2	3
3	2	1	2	1

(B) Partial initial state. This state could lead to different topological surfaces.

FIGURE 4. The initial states of a gradient can define the topological surface.

Definition 4.5. An initial state will be called monotone if the initial values in each row or column define a strictly monotonic sequence.

Definition 4.6. Let X be a topological surface. An initial state A will be called an initial state of X if the edges of A define the fundamental rectangle of X.

The following definition connects the visual gradient and topology.

Definition 4.7 (Initial topological gradient)**.** Given a topologocal surface X with a respective fundamental polygon gradient, an initial X state for a gradient matrix n is an initial state aligned with the fundamental polygon.

We are now ready to define visual gradients as respective topological surfaces. We start with the most intuitive one, a cylinder.

Definition 4.8. A cylinder gradient is a gradient that is defined by the fundamental polygon with only two parallel edges in the same direction, i.e., it represents a cylinder $\forall 1 \leq j \leq n : pixel(1, j) = pixel(n, j)$. A rotation of this gradient is of the same type.

(A) Cylinder (B) Cylinder (C) sphere

FIGURE 5. Examples of visual gradients for which the initial state defines a topological surface.

In Figs. 5a and 5b, we can see that the cylinder gradient is not unique, and this made us wonder whether we can determine an upper and lower bound to the number of gradients that can be obtained for a given topological surface.

We here give a glimpse of the power of the language we are trying to formulate.

Proposition 4.9. *Given a monotonic initial state for a given image $n \times n$ and $n \leq 256$, X must be a sphere.*

We omit the proof; for an example of a sphere gradient, see Fig. 5c.

4.2.4. *Pitfalls.* This research involves different disciplines, each of which requires a specialism. We may add more researchers from mathematics, computer science, and design to the team to ensure that progress is obtained.

4.2.5. *Expected outcomes and impact.* The expected outcome is a systematic math treatment for the visual gradient. This proposal has many different research outcomes. From the design point of view, understanding the mathematics can help the designer customize which visual gradient is best to apply in a given product. From the mathematical point of view, our formulation connects fields which are seen as very different from one another, e.g., geometry, topology, and combinatorics. In addition, there are hundreds of publications exploring Sudoku, and we believe that the scientific community in this field will be strongly interested in our language. Equally important, it will encourage mathematicians to open their mind and learn about diverse fields (in our case, the fields of design and art) that might not seem related to their main research. It may also be considered in the curriculum for undergraduate students in mathematics or design to demonstrate how intermediate mathematics can be applied with the right guidelines.

4.3. **Aim 3: Dynamical Tiling.** Usually, industrial design innovation is related to materials and artificial intelligence. When designers deal with mathematics, it usually relates to the geometric properties of the product, even though mathematics has a variety of tools that can be used, such as algebra, topology, etc. This research shows that innovation can come from unexpected places, such as group theory, which can define a dynamical tiling with different stages. This topic is essential if designers are to increase their toolbox not only in the final steps of defining a pattern but also in the initial steps of planning a product that has tiling properties.

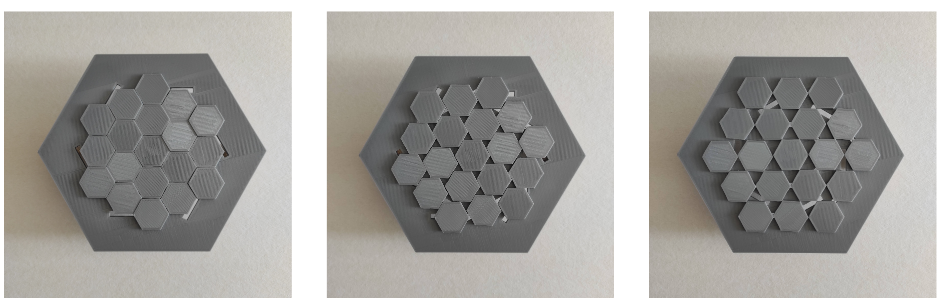
We intend to define a geodesic dome that is obtained by a tiling group, which can define a dynamical movement based on the respective sub-group. We may consider defining a proper mechanism that determines the movement, as has been done in [3]. Defined using mathematics, our dynamical tiling is not designed for a specific task. Instead, it is a concept that can be applied to the specific demand of a designer who intends to innovate a product with certain properties, such as reduction/expansion or exposing/concealing.

4.3.1. *Work plan.* In [6], the signatures of patterns and tilings and their respective group relationships have been defined. In this project, we intend to apply some of these relationships and define a dynamical movement between different stages, not just on a plane but also on spherical surfaces, especially geodesic domes. The mathematical exploration is based on the relations of spherical patterns (which are related to domes), i.e., the respective groups and sub-groups. To obtain the geodesic dome, we first consider a deforming plane model, as shown in Fig. 7, which we believe can give us the knowledge of how to define the respective mechanism.

4.3.2. *Preliminary results.* In [3], we discussed the relations among planar patterns, and show how a dynamical movement can be obtained between the hexagonal regular lattice which is defined by reflections, rotation, and translation (signature $*632$), and its sub-group, which is defined by rotation and translations (signature 632), as demonstrated in [1].

Before moving to spherical patterns, we first delve into planar patterns, since a planar pattern can be considered to be a local approximation of a spherical pattern. We need to decide which planar pattern and relation is best deform, not only according to mathematics but also according to material design.

In Fig. 7, we are given a local representation of a spherical pattern with the respective signature. Using this basic model, we attempt to evaluate what is the "optimal" movement with the help of the respective groups.



(A) Initial state, with signature $*632$

(B) Middle state, which leads to signature 632

(C) Open state, with signature $*632$

FIGURE 6. Our planar model defines, using a proper mechanism, a dynamical movement between tilings (made from PLA). When the mechanism reaches the end of the rail, it yields an extended pattern of signature $*632$.

In Fig. 8, planar patterns implemented with various materials are considered. In Fig. 8a, we consider paper and brass fasteners, which are flexible enough to obtain a deformation. We believe that this approach will give us another avenue in spherical pattern approximation to explore. In Fig. 8b, we use laser-cut wood and brass fasteners; this leads to a scissor-linkage mechanism for a pentagon that opens and closes. Since in some spherical patterns, pentagons play a major role, we believe that this kind of experiment can lead to the desired mechanism. In Fig. 8d, we combine the previous steps and attempt to approximate signature $*532$.

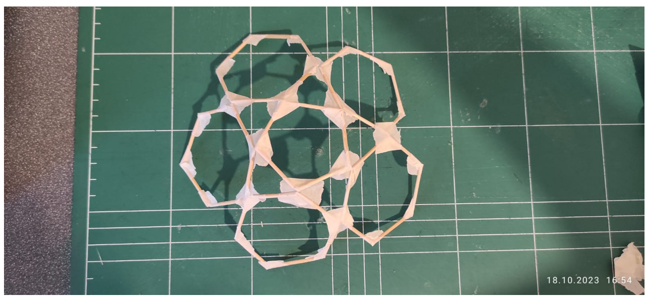
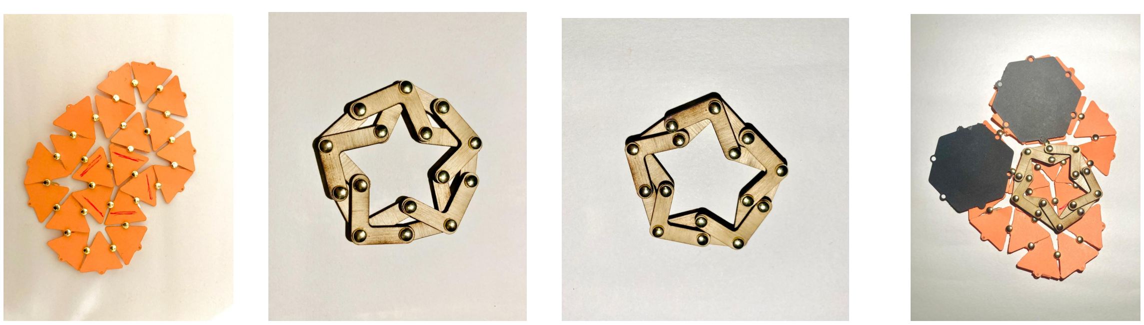


FIGURE 7. Our first naive experiment in a spherical pattern with signature $*532$. By exploring the object, e.g., removing the triangles, we attempt to predict the mechanism.



(A) (B) (C) (D)

FIGURE 8. Exploring movement, deformation, and mechanism.

4.3.3. *Pitfalls.* To develop a dynamical geodesic dome for real applications, the construction needs to be efficient. For example, if the mechanism is inside the dome, the space inside the dome will be reduced.

4.3.4. *Expected outcomes and impact.* This research will yield new mathematical tools for the average industrial designer. The designer will be able to choose a given pattern in which a movement is required in a physical object for a given purpose. Using our mathematical language, the pattern can be transformed into the respective fundamental area, tiling, and signature. Each such pattern has a sub-group that will be a natural candidate for the dynamical movement required.

5. Summary

In this project, I seek to find a symbiotic relationship between design and mathematics. In Section 4.1, I describe how industrial design based on mathematical foundations helps to give a tangible feel and look to music. Section 4.2 presents how design-based definitions can help formulate new ideas in mathematics. Finally, Section 4.3 discusses how abstract algebraic ideas can be adapted to industrial design.

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