Finding the maximum or minimum value of a two-variable quadratic function algebraically

Ali Othman - Al-Qasemi Academic College

In this article I present a simple method for finding the critical points (extrema) of a two-variable quadratic function. The method is a generalization of a method for finding the critical points of a one-variable quadratic function. The method for finding the critical points of a one-variable quadratic function depends exclusively on the idea of completing the square; that is, it is a purely algebraic method which can be taught to advanced students. As part of the solution presented here we address Hessian partial derivatives of the determinant, the case where the Hessian determinant is zero, and the necessary and sufficient conditions for determining whether or not a critical point is in fact an extrema in this case.

1. Analysis of a one-variable quadratic function

Finding the maximum or minimum value of a quadratic function with one variable

Let $f(x) = ax^2 + bx + c$ when $a \neq 0$. By applying square completion, we may represent the function equivalently as follows:

$$f(x) = \frac{1}{4a}(4a^2x^2 + 4bax + 4ac) = \frac{1}{4a}((2ax)^2 + 2b(2ax) + 4ac) = \frac{1}{4a}[(2ax + b)^2 - b^2 + 4ac]$$

$$= \frac{1}{4a}[(2ax + b)^2 - (b^2 - 4ac)]$$

We denote $\Delta = b^2 - 4ac$ (Δ is the discriminator).

therefore $f(x) = \frac{1}{4a}(2ax + b)^2 - \frac{\Delta}{4a}$.

In the case where $a > 0$, $f(x)$ takes on a minimum value when $2ax + b = 0$; that is, when $x = -\frac{b}{2a}$, and the minimum value is $-\frac{\Delta}{4a}$. The point $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$ is a vertex of the quadratic function and it is the lowest point.

In the case where $a < 0$, the function $f(x)$ takes on a maximum value when $2ax + b = 0$. That is, $x = -\frac{b}{2a}$ is a maximum, and the maximum value is $-\frac{\Delta}{4a}$. The point $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$ is the uppermost point.

In both cases, the point $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$ is called the vertex of the quadratic function.

A Hessian matrix is a quadratic matrix whose elements are the second order partial derivative of a function

$$ax^2 + bx + c = \frac{1}{4a}[(2ax + b)^2 - \Delta]$$

Note that: $f'(x) = 2ax + b$ (the derivative of the function).

Now let us generalize the idea of square completion to analysis of two-variable quadratic functions.

2. Two-variable quadratic functions

Let $f(x, y) = ax^2 + bxy + cy^2 + dx + ey + k$.

2.1 The basic case where $a \neq 0$ and $b \neq 0$ (or $c \neq 0$ and $b \neq 0$)

$$f(x,y) = \frac{1}{4a}(4a^2x^2 + 4abxy + 4acy^2) + dx + ey + k$$
$$= \frac{1}{4a}\left[(2ax + by)^2 - b^2y^2 + 4acy^2\right] + dx + ey + k$$
$$= \frac{1}{4a}\left[(2ax + by)^2 + 4adx + 4aey - \Delta y^2\right] + k$$

We distinguish among three cases:

2.1.1 Case I: $\Delta \neq 0$

where $\Delta = b^2 - 4ac$

Let us represent the function in a different way by applying square completion:

$$f(x, y) = \frac{1}{4a}[(2ax + by)^2 + 2d(2ax + by) + (4ae - 2bd)y - \Delta y^2] + k$$

$$= \frac{1}{4a}[(2ax + by + d)^2 - d^2 + (4ae - 2bd)y - \Delta y^2] + k$$

$$= \frac{1}{4a}\left[(2ax + by + d)^2 - \Delta\left(y^2 - 2\left(\frac{2ae - bd}{\Delta}\right)y + \frac{d^2}{\Delta}\right)\right] + k$$

$$= \frac{1}{4a}(2ax + by + d)^2 - \frac{\Delta}{4a}\left[(y - \frac{2ae - bd}{\Delta})^2 - (\frac{2ae - bd}{\Delta})^2 + \frac{d^2}{\Delta}\right] + k$$

We denote $y_0 = \frac{2ae-bd}{\Delta}$:

therefore:

$$f(x,y) = \frac{1}{4a}(2ax+by+d)^2 - \frac{\Delta}{4a}(y-y_0)^2 + \frac{\Delta \cdot {y_0}^2}{4a} - \frac{d^2}{4a} + k$$

If we apply the same procedure to the variable y (replacing y by x and conversely, replacing a by c and conversely, and replacing d by e and conversely, on the condition that $c \neq 0$), we obtain:

$$(1) \quad f(x,y) = \frac{1}{4c}(2cy+bx+e)^2 - \frac{\Delta}{4c}(x-x_0)^2 + \frac{\Delta \cdot {x_0}^2}{4c} - \frac{e^2}{4c} + k$$

where: $x_0 = \frac{2cd-be}{\Delta}$, $(\Delta = b^2 - 4ac)$.

x_0 , y_0 may be represented using determinants:

$$x_0 = \frac{\begin{vmatrix} -d & b \\ -e & 2c \end{vmatrix}}{\begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix}} \quad \text{and} \quad y_0 = \frac{\begin{vmatrix} 2a & -d \\ b & -e \end{vmatrix}}{\begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix}}$$

We see that (x_0, y_0) is the only solution of the system of equations:

$$\begin{cases} 2ax + by + d = 0 \\ bx + 2cy + e = 0 \end{cases}$$

(by Cramer's rule for solving a system of equations using determinants)

From equation (1) we obtain:

$$f(x_0,y_0) = \frac{(2ae-bd)^2}{4a\Delta} - \frac{d^2}{4a} + k$$

And also $f(x_0, y_0) = \frac{(2cd-be)^2}{4c\Delta} - \frac{e^2}{4c} + k$ (it's easy to check that the equality holds), therefore:

$$a \neq 0, \quad f(x,y) = \frac{1}{4a}(2ax+by+d)^2 - \frac{\Delta}{4a}(y-y_0)^2 + f(x_0,y_0)$$

In another form:

$$c \neq 0, \quad f(x,y) = \frac{1}{4c}(2cy+bx+e)^2 - \frac{\Delta}{4c}(x-x_0)^2 + f(x_0,y_0)$$

When $(x_0, y_0)$is the only solution of the system of equations:

$$\begin{cases} 2ax + by + d = 0 \\ 2cy + bx + e = 0 \end{cases}$$

we may also obtain the following representation when $ac \neq 0$:

$$f(x,y) = \frac{1}{8a}[(2ax + by + d)^2 - \Delta(y - y_0)^2] + \frac{1}{8c}[(2cy + bx + e)^2 - \Delta(x - x_0)^2] + f(x_0, y_0)$$

Now it is easy to determine the maximum or minimum in this case:

I a) if $\Delta < 0$ and $a > 0$ (then $c > 0$ too) then clearly $f(x,y) \geq f(x_0, y_0)$ for all (x, y).

Therefore (x_0, y_0) is the absolute minimum of the function.

If $\Delta < 0$ and $a < 0$ (then $c < 0$ too), then $f(x,y) \leq f(x_0, y_0)$ for all (x, y).

Therefore (x_0, y_0) is the absolute maximum of the function.

I b) In the case $\Delta > 0$ and $a > 0$, let us look at the following representation:

$$f(x,y) = \frac{1}{4a}(2ax + by + d)^2 - \frac{\Delta}{4a}(y - y_0)^2 + f(x_0, y_0)$$

If (x, y) is on the line $2ax + by + d = 0$ then $f(x,y) = -\frac{\Delta}{4a}(y - y_0)^2 + f(x_0, y_0)$ and approaches $-\infty$ when $|y|$ approaches ∞. Likewise, for each x, $f(x, y_0) = \frac{1}{4a}(2ax + by_0 + d)^2 + f(x_0, y_0)$ holds and approaches ∞ when $|x|$ approaches ∞. Therefore (x_0, y_0) is neither a maximum nor a minimum of the function.

And similarly for $\Delta > 0$ and $a < 0$.

2.1.2 Case II: $\boldsymbol{\Delta = 0}$. In this case:

$$f(x,y) = \frac{1}{4a}[(2ax + by)^2 + 4adx + 4aey] + k = \frac{1}{4a}[(2ax + by)^2 + 2d(2ax + by) - 2bdy + 4aey] + k$$

$$= \frac{1}{4a}[(2ax + by + d)^2 - d^2 + 2(2ae - bd)y] + k$$

$$= \frac{1}{4a}(2ax + by + d)^2 + \frac{1}{2a}(2ae - bd)y + k - \frac{d^2}{4a}$$

If $\Delta = 0$ and $a \neq 0$ then:

$$f(x,y) = \frac{1}{4a}(2ax + by + d)^2 + \frac{1}{2a}(2ae - bd)y + k - \frac{d^2}{4a}$$

We distinguish between two cases:

II a) if $2ae - bd = 0$, then:

$$f(x,y) = \frac{1}{4a}(2ax + by + d)^2 + k - \frac{d^2}{4a}$$

If $a > 0$ then the function takes on its minimum value at all points that lie on the line $2ax + by + d = 0$, and the minimum value is $k - \frac{d^2}{4a}$.

And if $a < 0$, then the function takes on its maximum value at all points that lie on the line $2ax + by + d = 0$, and the maximum value is $k - \frac{d^2}{4a}$.

II b) In the case that $\Delta = 0$ and $2ae - bd \neq 0$, the function is:

$$f(x,y) = \frac{1}{4a}(2ax + by + d)^2 + \frac{1}{2a}(2ae - bd)y + k - \frac{d^2}{4a}$$

The function has neither an absolute maximum nor an absolute minimum, since along the line $2ax + by + d = h$ for every point (x_1, y_1) if $2ax_1 + by_1 + d = h$, then:

$$f(x_1,y_1) = \frac{1}{4a}h^2 + \frac{1}{2a}(2ae - bd)y_1 + k - \frac{d^2}{4a}$$

Therefore if (x', y') and (x'', y'') are on the line $2ax + by + d = h$ on either side of the point $(x_1 \ , y_1)$

then certainly one of $f(x', y')$ and $f(x'', y'')$ is greater than $f(x_1 \ , y_1)$ and the other is less than it.

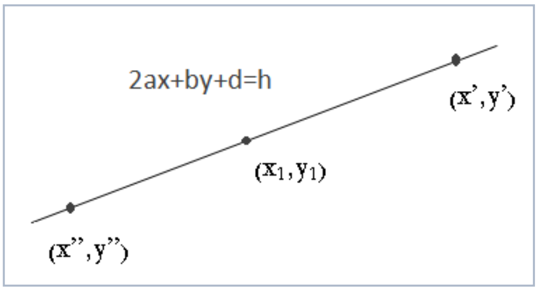

Figure 1

Therefore (x_1, y_1) is neither a local maximum nor a local minimum of the function.

Note: We'd like to point out here that both expressions $2ax + by + d$ and $bx + 2cy + e$ are partial derivatives of the function: $f(x, y)$

That is :

$$\frac{\partial f}{\partial x} = 2ax + by + d$$

$$\frac{\partial f}{\partial y} = bx + 2cy + e$$

The following determinant: $\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$ is called **Hessian**, and in this case is equal at point (x_0, y_0)

to:
$$\begin{vmatrix} 2a & b \\ b & 2c \end{vmatrix} = 4ac - b^2 = -\Delta$$

In advanced mathematics, the significance of partial derivatives in finding local maxima and minima is well known. We arrived at the use of partial derivatives through square completion

Example 1: Given the function:

$$f(x, y) = 4x^2 + 7xy + 5y^2 - 4x + 12y$$

Check whether the function has an absolute maximum or minimum, if so find it, and find at what point it exists:

a) by the rule we found in this article.

b) directly (in order to understand the method in finding the rule).

Solution:

a) solution by rule: $\Delta = 7^2 - 4 \cdot 4 \cdot 5 = -31 < 0$ and $a = 4 > 0$. Therefore the function has an absolute minimum. In order to find the absolute minimum value let us solve the following system of equations:

$$\begin{cases} 8x + 7y - 4 = 0 \\ 7x + 10y + 12 = 0 \end{cases}$$

The solution is $(x_0 \ , y_0) = (4, -4)$, and therefore f_{min}.

b) direct solution (for understanding the procedure):

$$f(x,y) = \frac{1}{4}(16x^2 + 28xy + 20y^2) - 4x + 12y$$

$$= \frac{1}{4}\left[(4x + \frac{7}{2}y)^2 - (\frac{7}{2}y)^2 + 20y^2\right] - 4x + 12y$$

$$= \frac{1}{4}\left[(4x + \frac{7}{2}y)^2 + \frac{31}{4}y^2\right] - 4x + 12y$$

$$= \frac{1}{4}\left[(4x + \frac{7}{2}y)^2 - 16x + 48y\right] + \frac{31}{16}y^2$$

$$= \frac{1}{4}\left[(4x + \frac{7}{2}y)^2 - 4(4x + \frac{7}{2}y) + 62y\right] + \frac{31}{16}y^2$$

$$= \frac{1}{4}\left[(4x + \frac{7}{2}y)^2 - 4(4x + \frac{7}{2}y)\right] + \frac{31}{16}y^2 + \frac{31}{2}y$$

$$= \frac{1}{4}\left[(4x + \frac{7}{2}y - 2)^2 - 4\right] + \frac{31}{16}(y^2 + 8y)$$

$$= \frac{1}{4}(4x + \frac{7}{2}y - 2)^2 + \frac{31}{16}(y + 4)^2 - 31 - 1$$

$$= \frac{1}{16}(8x + 7y - 4)^2 + \frac{31}{16}(y + 4)^2 - 32$$

Clearly the function has an absolute minimum when both squares are equal to 0 . This holds when $y_0 = -4$ and $x_0 = 4$.

The minimum value is -32.

Example 2: Analyze the following function:

$$f(x,y) = x^2 + 6xy + 9y^2 - 8x - 12y$$

a) by the rule: $\Delta = 6^2 - 4 \cdot 1 \cdot 0 = 0$

$$2ae - bd = 2 \cdot 1 \cdot (-12) - 6 \cdot (-8) = 24 \neq 0$$

The function has neither absolute nor relative minima or maxima.

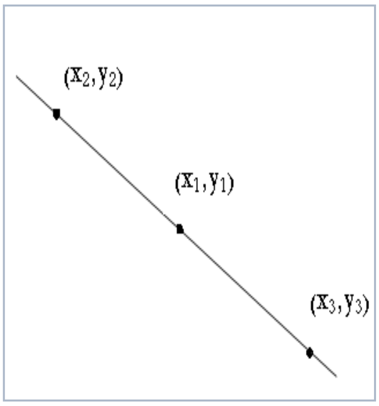

Figure 2

b) direct solution:

$$f(x,y) = (x+3y)^2 - 8(x+3y) + 12y = (x+3y-4)^2 + 12y - 16$$

$$f(4-3y,y) = 0^2 + 12y - 16$$

When y approaches ∞, the function values approach ∞.

When y approach $-\infty$, the function values approach $-\infty$.

Therefore the function has no absolute minima or maxima. We will now show that the function has no relative minima or maxima.

If (x_1, y_1) is a point, and $x_1 + 3y_1 - 4 = h$, then (x_2, y_2) and (x_3, y_3) are two points around the point (x_1, y_1) on either side of (x_1, y_1) .

Therefore $f(x_2, y_2) = 12y_2 - 16$ and $f(x_3, y_3) = 12y_3 - 16$.

If $y_2 > y_1 > y_3$, then $f(x_2, y_2) > f(x_1, y_1) > f(x_3, y_3)$. Therefore (x_1, y_1) is not a local minimum.

2.2 Survey of the other cases:

2.2.1: When $b = 0$

2.2.1.1: When $b = 0$ and $a > 0$ and $c > 0$.

then:

$$f(x,y) = ax^2 + cy^2 + dx + ey + k = a\left(x + \frac{d}{2a}\right)^2 + c\left(y + \frac{e}{2c}\right)^2 + \frac{4kac - cd^2 - ae^2}{4ac}$$

then the point $\left(-\frac{d}{2a}, -\frac{e}{2c}\right)$ is the absolute minimum of the function, and the minimum value is $\frac{4kac-cd^2-ae^2}{4ac}$.

When $b = 0$ and $a < 0$ and $c < 0$, $f(x,y) = a\left(x + \frac{d}{2a}\right)^2 + c\left(y + \frac{e}{2c}\right)^2 + \frac{4kac - cd^2 - ae^2}{4ac}$.

Clearly the point $\left(-\frac{d}{2a}, -\frac{e}{2c}\right)$ is a maximum of the function, and the maximum value is $\frac{4kac-cd^2-ae^2}{4ac}$.

2.2.1.3: When $b = 0$ and $ac < 0$. in this case the values of the function will approach ∞ and $-\infty$. Therefore the function does not take on a maximum nor a minimum.

2.2.2: When $b \neq 0$ and $a = 0$ and also $c = 0$:

In this case it is easy to see that the function values approach both ∞ and $-\infty$. Therefore the function does not take on a maximum nor a minimum.

For further reading on maxima and minima of two-variable functions, see, for example:

https://www.math.ubc.ca/~feldman/m105/maxmin.pdf

The author: Dr. Ali Othman, lecturer of mathematics at Al-Qasemi Academic College, Baqa al-Gharbiyye, Israel