J. Fluid Mech. (2020), *vol.* 11, *pp.* 1–000. Preprint prepared by the Authors

DOI: 10.1017/S000000000000000X

Integral-equation approach to resonances in circular two-layer flows around an island with bottom topography

Department of Physics, Bar Ilan University, Ramat-Gan 52900, Israel

(Received 1 February 2020)

~~A~~This paper presents an integral-equation approach to the linear instability problem of two-layer quasi-geostrophic flows around ~~a~~ circular islands with bottom topography ~~is studied~~. The study extends an earlier barotropic model of similar geometry and topography and focuses on the degree to which the topographic waves in the lower layer resonate with the basic flow in each layer. The integral approach poses the instability problem in a physically elucidating way, ~~in which~~whereby the resonating neutral waves in the system ~~can be~~are directly identified ~~directly~~. The flows investigated are composed of uniform potential-vorticity (PV) rings in each layer, ~~having~~with the PV of each ring being of opposite sign~~s~~. Four types of instabilities are identified: instability caused by the resonance of ~~the~~ Rossby waves traveling along the liquid contours at the edge of each PV ring (CC resonance), instability caused by the resonance of the wave at the upper-layer contour and the topographic waves outside the lower-layer contour (C_1T), a similar resonance of the lower-layer contour with the topographic waves (C_2T), and a resonance between one eigenmode of the contour~~s~~ subsystem with the topographic waves (CCT). The three la~~st~~tter resonances lead to ~~a~~ critical ~~level~~layer instabilities and can be identified as resonances ~~of~~between the contour waves ~~with~~and a collection of singular topographic modes ~~having~~with a critical layer. The C_1T (C_2T) instability occurs when the lower-layer ring is sufficiently thin ~~enough~~ and the basic flow travels counterclockwise (clockwise). The neutral PV perturbations in the outer region behave asymptotically ~~behave~~ as barotropic (BT) or baroclinic (BC) modes that, when traveling clockwise, have spiral shapes and are wavelike in the radial direction. Usually, the BT mode is the ~~one~~mode ~~that~~in resonance with the contours~~,~~ but, ~~in case of~~for small growth rates, the BC mode may be the dominant ~~one~~mode. The nonlinear evolution of the CC resonance usually leads to emission of dipolar modons ~~which~~that then return to the island and are re-emitted in a quasi~~-~~periodic manner. The contour~~s~~-topography instabilities may produce a narrow PV ring at the lower layer at the location of the critical layers of the dominant resonating topographic perturbations; this ring interacts with the original rings to form a quasi-stationary structure (e.g., a tripole) that rotates counterclockwise for a relatively long time before splitting into emitted modons.

1. Introduction

Islands in the stratified ocean ~~might~~can be surrounded by ~~have~~ complex and variable current circulation patterns ~~around them~~ (**?**). Closed flows ~~which~~that are anomalous~~,~~ (i.e., ~~follow an~~flow in a direction opposite ~~direction to~~ the overall circulation of the surrounding ocean~~,~~) have been observed around Iceland, Taiwan, the islands of Kuril Chain (**?**) and the Pribilof ~~i~~Islands (**?**). In most cases, these anomalous circulations are anticyclonic (clockwise) in the northern hemisphere and are wind-driven. Waves generated in the vicinity of the islands may be trapped by the slop~~p~~ing topography or the coast~~,~~ and also contribute to the circulation strength; ~~the~~ trapping by islands was shown for barotropic (BT) flows (see, e.g., Refs. ???) ~~as well as~~and for the stratified sea (see, e.g., Refs. ???). In this paper, we study the conditions for instability of such baroclinic (BC) flows in an idealized model, where the island is circular and the bottom topography is conical (i.e., the beta~~-~~-cone model).

To account for the stratification of the ocean, we use the simplified two-layer quasi-geostrophic (QG) ~~simplified~~ model (?). The linear ~~baroclinic~~BC instability problem was solved by ? for the case of two-layer zonal uniform QG flow over a flat bottom~~ by ?,~~ and later solved by ? with ~~inc~~the ~~luding~~ bottom topography included~~by ?~~. For circularly symmetric flows, the instability problem was investigated by ? for ~~baroclinic~~BC QG vortices with a flat bottom and continuous stratification. Circular ~~baroclinic~~BC two-layer flows were investigated by ?~~,~~ for flows confined to an annular channel. Our model ~~is~~ differ~~ent~~s in several respects~~.~~: First, ~~in our model~~it ~~there is~~requires no external boundary. Second, the basic flow ~~is~~ differ~~ent~~s, as described below; ~~especially~~in particular, ~~in our model the~~ currents in the two layers may ~~be~~flow in opposite ~~in~~ directions in our model. Related to this is the fact that, in our case, the ratio of the bottom slope to the basic isopycnal slope is not constant, ~~and therefore~~so this ratio plays no fundamental role.

~~T~~? investigated the instability associated with idealized circularly~~-~~ symmetric ~~barotropic~~BT currents around circular islands with bottom topography~~ was investigated by ?~~. There the flow was composed of two constant~~s PV~~potential-vorticity (constant-PV) rings around the island and the velocity outside the rings was zero. The purpose of this paper is to consider a variant of that model appropriate for a two-layer flow; now any layer consists of ~~one~~a single constant-PV ring. The flows in the two layers may have opposite directions and the velocity outside the rings does not vanish identically, but rather only the ~~barotropic~~BT velocity. ~~f~~Figure 1 presents schematically the velocities and PV~~s~~ profiles.

The concept of resonance provides a physical interpretation of instabilities in two-layer shallow~~-~~-water flows~~ is made possible using the concept of resonance~~. As ~~has been~~ shown by many authors, different types of instabilities can be identified as resonances between neutral waves; the type of ~~the~~ instability is determined by the interacting waves. The resonance ~~is~~ usually ~~seen by~~occurs at the ~~crossing~~intersection of the dispersion curves [i.e., ~~of~~the curve of the phase velocity v~~s.~~ersus the wave number of the neutral modes (?)]. This was demonstrated in zonal shear flows (???) and zonal two-layer flows (????). In all these papers the resonan~~t~~ce viewpoint was only applied to the case of shallow-water systems without the QG approximation; in this paper it is also applied ~~also~~ to QG flows.

~~To~~We identify the resonances in a simple way~~,~~ by using an integral-equation approach to the linear instability problem~~ is undertaken~~. To date, the integral approach ~~was used so far~~has only been used for ~~barotropic~~BT flows by ?, with no further ~~development~~extension elsewhere to ~~baroclinic~~BC flow~~s~~. ~~t~~This paper fills this gap~~. It is~~ by show~~n~~ing that the integral approach poses the instability problem in a physically elucidating way~~, in which~~whereby the coupling between the various wave types ~~can be~~is directly identified~~ directly~~.

The basic flow considered ~~in this paper~~herein ~~is~~differs fundamentally ~~different~~ from the ~~barotropic~~BT case studied by ? ~~since here~~because, here, the basic velocity is zero outside the rings~~ the basic velocity~~ ~~at~~in the lower layer~~ is nonzero~~, as ~~well a~~is the PV gradient (see ~~f~~Figure 1). This fact ~~makes possible~~permits the existence of singular neutral perturbations whose phase velocity ~~is~~ equals ~~to~~ the basic velocity at some ~~place~~distance [i.e., ~~having~~the lower layer has a critical layer (??)]. ~~It is known that i~~ If a linear stability analysis shows that critical-layer eigenfunctions are neutrally stable, a more careful analysis of the initial-value problem would show~~s~~ that ~~they~~the eigenfunctions actually ~~cause~~lead asymptotically over time to an algebraic time dependence~~ asymptotically with time~~ (??).**[AU: Please verify in particular all red text to ensure that the intended**

Figure 1: Schematic profiles of the basic velocity in the upper layer $\bar{V}_1$ (solid, red online)~~,~~ and in the lower layer $\bar{V}_2$ (dashed, red online), and the PV in the upper layer Q_1 (solid, blue online) and in the lower layer Q_2 (dashed, blue online).

meaning is maintained.]The time dependence is found mathematically from the singularities of the modes on the complex-frequency plane. Here on the beta cone ~~it~~ is shown that new singularities, not present in the zonal case, appear; their damping effect is ~~analytically~~ calculated analytically.

~~C~~A critical layer instability (**?**) was observed experimentally by **?** for a columnar vortex in stratified fluid and was studied in shallow-water ~~one-layer~~single-layer flows (**?**). **?** ~~have~~ show~~n~~ed that this instability could be interpreted as the resonance ~~of~~between a nonsingular mode ~~with~~and a collection of singular modes. For the basic flow considered in this paper, resonances involving the topographic waves at the lower layer lead to a critical-layer instability. The resonating perturbations in this case are identified, and ~~its~~their effect on the nonlinear evolution of the flow is studied numerically.

The outline of this paper is as follows: ~~In~~ §2 ~~we~~ present the basic equations of the model of quasigeostrophic two-layer flows, and ~~in~~ §3 ~~we~~ gives the derivation of the integral eigenvalue equation of the linear stability analysis. ~~In~~ §4 ~~we~~ appl~~y~~ies the integral equation to the basic flow composed of the two-layer PV rings ~~schematically~~ plotted schematically in ~~f~~Figure 1. The resonance viewpoint is then presented for this flow in §5. ~~In~~ §6 discusses the spectrum of solutions in the exterior region $r > R_2$ ~~is discussed~~. These neutrally stable solutions (according to ~~the~~a linear stability analysis) ~~are the ones that~~ may resonate with the waves at the contours of the PV discontinuities at R_1 and R_2, ~~their~~and their time-dependent damping ~~over time~~ is also found. ~~In~~ §7 further explores the resonances ~~are further explored~~via ~~by the~~ dispersion curves, calculates the growth rates and ~~the~~ structure of ~~the~~ unstable perturbations ~~are calculated~~, and discusses the conditions for the dominance of ~~barotropic~~BT versus ~~baroclinic~~BC couplings ~~are discussed~~. ~~In~~Finally, §8 examines ~~the influence of~~how instability type ~~on~~affects the nonlinear evolution of the flow ~~is examined~~.

2. Two~~-~~-layer flows on a beta cone: Governing equations

Consider a two-layer ~~quasigeostrophic~~ QG model in which the flow surrounds a cylindrical island. The bottom ~~outside~~around the island is assumed to have a constant radial slope so that the depth increases linearly ~~offshore~~with distance from the island. Under the ~~quasigeostrophic~~QG approximation and the rigid-lid condition at the sea surface, the flow is effectively two~~-~~-dimensional in each layer. The variables of the upper and lower layers are denoted by the subscripts 1 and 2, respectively. The unperturbed layer thickness is denoted ~~by~~ H_i (~~$i=1$~~$i=1,2$), and the~~ir~~ sum of layers by H. In the polar coordinates r and θ, the radial and azimuthal components of the velocity, u_i and v_i, respectively, in each layer ($i=1,2$)~~,~~ can be expressed in terms of a streamfunction Ψ_i ~~via the equations~~by

$$u_i = -\frac{1}{r}\frac{\partial \Psi_i}{\partial \theta}, \quad v_i = \frac{\partial \Psi_i}{\partial r}. \tag{2.1}$$

In the following, whenever the subscript i appears, it refers to ~~the~~layer i~~-th layer~~. The slope at the bottom introduces a linear term in r for the PV at the lower layer (see **?** for details). The proportionality constant is the topographic beta, $\beta = -f\tan(\alpha)/H_2$, where f is the Coriolis parameter. ~~It is assumed that t~~The island~~'s size~~ is assumed to be small compared ~~to~~with the planetary scale, so f may be regarded as ~~being~~ constant (this is analogous to the f-plane approximation, cf. ?). For an island in the northern hemisphere, β is negative.

In terms of the streamfunctions, the PVs in layers 1 and 2 are defined as (cf. ?)

$$Q_1 = \nabla^2\Psi_1 - \frac{f^2}{g'H_1}(\Psi_1 - \Psi_2), \quad Q_2 = \nabla^2\Psi_2 + \frac{f^2}{g'H_2}(\Psi_1 - \Psi_2) + \beta r, \tag{2.2}$$

where $g' = g(\rho_2 - \rho_1)/\rho_1$ is the reduced gravity (g being the acceleration due to gravit~~ational acceleration~~y, and ρ_1 and ρ_2 being the layer densities).

On the beta cone, a natural length scale is the radius R of the island. We are interested in flows whose horizontal length scale is R, such that the curvature plays a dominant role, so $r \approx R$. Flows having much smaller length scale behave locally as straight flows, ~~while~~whereas at much larger length scales the island's influence is negligible. In §3 the basic flow is defined, where the PV in the upper layer is constant inside a ring, Γ_1. This determines ~~a~~the timescale ~~for the time,~~ $1/|\Gamma_1|$. Therefore, assuming that ~~the~~ time scales advectively, we transform the variables into non~~-~~dimensional variables via

$$t \to t/|\Gamma_1|, \quad r \to Rr, \quad Q_i \to |\Gamma_1|Q_i, \quad \Psi_i \to |\Gamma_1|R^2\Psi_i, \quad \beta \to |\Gamma_1|\beta/R. \tag{2.3}$$

Nondimensionalization of equations (2.2) then yields

$$Q_1 = \nabla^2\Psi_1 - \frac{\Lambda^2}{\lambda_1}(\Psi_1 - \Psi_2), \quad Q_2 = \nabla^2\Psi_2 + \frac{\Lambda^2}{\lambda_2}(\Psi_1 - \Psi_2) + \beta r, \tag{2.4}$$

where $\Lambda^2 = (R/L_{Ro})^2$ is the reverse Burger number, and ~~$L_{Ro} = \sqrt{g'H/f_0^2}$~~ $L_{Ro} = (g'H/f_0^2)^{1/2}$ is the Rossby deformation radius. In the ocean, L_{Ro} varies from about 1 km at high latitudes to about 400 km at the equator (?). Small islands may have a radius of few kilometers, ~~while~~whereas large ~~ones~~islands may ~~reach~~have a radius of 200 km. Therefore, Λ may change from 10^{-4} to 200. ~~In order t~~To be consistent with the ~~quasigeostophic~~QG approximation ~~mentioned~~presented above, Λ should be of order ~~1~~unity or less (?). Therefore, ~~mostly~~ we mostly use $\Lambda = 1$; ~~T~~this means~~,~~ that the island's size is of the same order of magnitude as the Rossby deformation radius. The relative thickness of each layer i is denoted ~~by~~ $\lambda_i = H_i/H$ $(i = 1, 2)$, with H being the total thickness of the fluid~~;~~: $H = H_1 + H_2$.

The PV conservation equations governing the dynamics are

$$\frac{\partial Q_i}{\partial t} + \frac{1}{r}\left(\frac{\partial \Psi_i}{\partial r}\frac{\partial Q_i}{\partial \theta} - \frac{\partial \Psi_i}{\partial \theta}\frac{\partial Q_i}{\partial r}\right) = 0 \ (i = 1, 2). \tag{2.5}$$

3. The integral eigenvalue equations

We represent the PVs Q_1 and Q_2 and the streamfunctions Ψ_1 and Ψ_2 of the flow as sums of the basic-state values (indicated by a bar) and the associated perturbations,

$$Q_i = \bar{Q}_i(r) + q_i(r, \theta, t), \quad \Psi_i = \bar{\Psi}_i(r) + \psi_i(r, \theta, t). \tag{3.1}$$

~~Under the assumption of~~Assuming small perturbations, the linearized equations expressing conservation of PV ~~conservation equations~~ that result from (2.1) and (2.5) are

$$\frac{\partial q_1}{\partial t} + \frac{\bar{V}_1}{r}\frac{\partial q_1}{\partial \theta} - \frac{1}{r}\frac{\partial \psi_1}{\partial \theta}\frac{d\bar{Q}_1}{dr} = 0, \quad \frac{\partial q_2}{\partial t} + \frac{\bar{V}_2}{r}\frac{\partial q_2}{\partial \theta} - \frac{1}{r}\frac{\partial \psi_2}{\partial \theta}\frac{d\bar{Q}_2}{dr} = 0. \tag{3.2}$$

The perturbations are ~~thought of as~~considered to be associated with an azimuthal integer mode number m and (generally complex) frequency ω:

$$\{q_i(r, \theta, t), \ \psi_i(r, \theta, t)\} = \{\mathcal{Q}_i(r), \ \Phi_i(r)\}e^{i(m\theta - \omega t)}, \tag{3.3}$$

where we suppress the explicit notation of m in $\mathcal{Q}_i(r)$ and $\Phi_i(r)$ to keepsimplify the notation easier; theeis notation m is also dropped also in subsequent expressions. Using (3.3) in (3.2) yields the Rayleigh equations,

$$\left(\frac{\bar{V}_i(r)}{r} - \frac{\omega}{m}\right)\mathcal{Q}_i - \frac{\Phi_i}{r}\frac{d\bar{Q}_i}{dr} = 0. \tag{3.4}$$

By using (2.4) and (3.3), the functions $\mathcal{Q}_i(r)$ and $\Phi_i(r)$ are related via the equations

$$\mathcal{Q}_1 = \frac{d^2\Phi_1}{dr^2} + \frac{1}{r}\frac{d\Phi_1}{dr} - \frac{m^2}{r^2}\Phi_1 - \frac{\Lambda^2}{\lambda_1}(\Phi_1 - \Phi_2), \tag{3.5}$$

$$\mathcal{Q}_2 = \frac{d^2\Phi_2}{dr^2} + \frac{1}{r}\frac{d\Phi_2}{dr} - \frac{m^2}{r^2}\Phi_2 + \frac{\Lambda^2}{\lambda_2}(\Phi_1 - \Phi_2). \tag{3.6}$$

Given $\mathcal{Q}_1$ and $\mathcal{Q}_2$, the equations (3.5) and (2) for the streamfunctions can be decoupled. TBecause the term βr is absent, the decoupling is possible here, in contrast towith its impossibility in equations (2.4) where it is not possible since now the term βr is absent; this is because equations (3.5) and (2) deal with the perturbations of the PVs. Consider the barotropic (BT) and baroclinic (BC stream-function perturbations,

$$\Phi_{\rm BT} = \lambda_1\Phi_1 + \lambda_2\Phi_2, \quad \Phi_{\rm BC} = \Phi_1 - \Phi_2, \tag{3.7}$$

and the corresponding PV perturbations,

$$\mathcal{Q}_{\rm BT} = \lambda_1\mathcal{Q}_1 + \lambda_2\mathcal{Q}_2, \quad \mathcal{Q}_{\rm BC} = \mathcal{Q}_1 - \mathcal{Q}_2. \tag{3.8}$$

From equations (3.5) and (2) and the definitions (3.7) and (3.8), we get the equations

$$\frac{d^2\Phi_{\rm BT}}{dr^2} + \frac{1}{r}\frac{d\Phi_{\rm BT}}{dr} - \frac{m^2}{r^2}\Phi_{\rm BT} = \mathcal{Q}_{\rm BT}, \tag{3.9}$$

$$\frac{d^2\Phi_{\rm BC}}{dr^2} + \frac{1}{r}\frac{d\Phi_{\rm BC}}{dr} - \frac{m^2}{r^2}\Phi_{\rm BC} - \frac{\Lambda^2}{\lambda_1\lambda_2}\Phi_{\rm BC} = \mathcal{Q}_{\rm BC}, \tag{3.10}$$

where the relation $\lambda_1 + \lambda_2 = 1$ was used in the last equation.

The general solutions to (3.9) and (3.10) can be written as

$$\Phi_{\rm BT}(r) = \int_R^\infty G_{\rm BT}(r,r')\mathcal{Q}_{\rm BT}(r')dr', \quad \Phi_{\rm BC}(r) = \int_R^\infty G_{\rm BC}(r,r')\mathcal{Q}_{\rm BC}(r')dr', \tag{3.11}$$

where $G_{\rm BT}(r,r')$ and $G_{\rm BC}(r,r')$ are the barotropicBT and baroclinicBC Green's functions, respectively. The derivations and expressions for these Green's functions appear in Appendix B. From (3.7) we get the expression of the streamfunction in each layer in terms of the barotropicBT and baroclinicBC modes,

$$\Phi_1 = \Phi_{\rm BT} + \lambda_2\Phi_{\rm BC}, \quad \Phi_2 = \Phi_{\rm BT} - \lambda_1\Phi_{\rm BC}. \tag{3.12}$$

Using (3.8), (3.11), and (3.12) we get

$$\Phi_1(r) = \int_R^\infty [G_{11}(r,r')\mathcal{Q}_1(r') + G_{12}(r,r')\mathcal{Q}_2(r')]dr', \tag{3.13}$$

$$\Phi_2(r) = \int_R^\infty [G_{21}(r,r')\mathcal{Q}_1(r') + G_{22}(r,r')\mathcal{Q}_2(r')]dr', \tag{3.14}$$

where the four Green's functions G_{ij} $(i,j=1,2)$ are defined as

$$G_{11}(r,r') = \lambda_1 G_{\mathrm{BT}}(r,r') + \lambda_2 G_{\mathrm{BC}}(r,r'), \tag{3.15}$$

$$G_{12}(r,r') = \lambda_2 [G_{\mathrm{BT}}(r,r') - G_{\mathrm{BC}}(r,r')], \tag{3.16}$$

$$G_{21}(r,r') = \lambda_1 [G_{\mathrm{BT}}(r,r') - G_{\mathrm{BC}}(r,r')], \tag{3.17}$$

$$G_{22}(r,r') = \lambda_2 G_{\mathrm{BT}}(r,r') + \lambda_1 G_{\mathrm{BC}}(r,r'). \tag{3.18}$$

~~By~~Based on (3.13) and (3.14), the function G_{ij} is the Green's function that connects a PV perturbation ~~at~~in ~~the~~layer j~~th layer~~ to the streamfunction ~~at~~in ~~the~~layer i~~th layer~~. Note also that the no-slip boundary condition ~~of no-slip~~ at the cylindrical wall [i.e., $\Phi_1(R)=\Phi_2(R)=0$ by ~~equations~~ (2.1) and (3.3)] is satisfied automatically by ~~equations~~ (3.13) and (3.14)~~ automatically~~. ~~Now w~~We now express the streamfunctions in terms of the PV perturbations by inserting (3.13) and (3.14) into (3.4) ~~and~~to get

$$\frac{m\bar{V}_i(r)}{r}\mathcal{Q}_i - \frac{m}{r}\frac{d\bar{Q}_i(r)}{dr}\int_R^\infty [G_{i1}(r,r')\mathcal{Q}_1 + G_{i2}(r,r')\mathcal{Q}_2]\,dr' = \omega\mathcal{Q}_i(r). \tag{3.19}$$

Equation (3.19) constitutes a system of two linear integral equations for the PV perturbations at both layers. In the next section, we apply these equations ~~for~~to flows composed of two-layer constant-PV rings.

4. Flows composed of two-layer rings

4.1. *Basic flow profile*

As stated above, ~~for a basic state~~ in the subsequent stability analysis, we take as a basic state a circularly symmetric flow composed of a uniform-PV ring in each layer. The ring in the upper (lower) layer is bounded by ~~the~~a rigid contour at $r=R$ and ~~the~~a material contour at $r=R_1$ (R_2), the latter of which we denote ~~by~~ C_1 (C_2). ~~Similarly, the ring in the lower layer is bounded by the rigid contour at $r=R$ and the material contour at $r=R_2$, which we denote by C_2.~~ Outside the rings, the PV of each layer equals the background PV. Denoting by $\bar{Q}_i$ the PV of the basic flow~~ by $\bar{Q}_i$,~~ and by Γ_1 and Γ_2 the PV in the upper and lower rings~~ by Γ_1 and Γ_2~~, respectively, we write

$$\bar{Q}_1(r) = \begin{cases} \Gamma_1, & R \leqslant r \leqslant R_1 \\ 0, & R_1 < r, \end{cases} \qquad \bar{Q}_2(r) = \begin{cases} \Gamma_2, & R \leqslant r \leqslant R_2 \\ \beta r, & R_2 < r. \end{cases} \tag{4.1}$$

The PV-jumps across each contour are

$$\Delta_1 = -\Gamma_1, \quad \Delta_2 = \beta R_2 - \Gamma_2. \tag{4.2}$$

The expressions for the basic streamfunctions $\bar{\Psi}_i$ and velocities $\bar{V}_i$ resulting from this PV~~s~~ configuration are derived in Appendix A. Since the flow is attached to a rigid cylindrical wall (the island), a natural (although ~~not~~ unnecessary) boundary condition would be the no-slip condition~~,~~ (i.e., the vanishing of the velocity at $r=R$). This condition results from the role of turbulent viscosity in the vicinity of the vertical wall during the formation of the closed flow, as explained in detail in ~~the paper by~~Ref. **?**. The vanishing of the velocity at the rigid boundary at $r=R$ imposes ~~a~~the following relation between Γ_1 and Γ_2 (see Appendix A for details)~~,~~:

$$\Gamma_2 = \frac{-2R^3\beta\lambda_2 + 2R_2^3\beta\lambda_2 + 3\Gamma_1 R^2\lambda_1 - 3\Gamma_1 R_1^2\lambda_1}{3\lambda_2(R_2^2 - R^2)}. \tag{4.3}$$

~~S~~Figure 1 shows schematic profiles of the velocities and PVs ~~at~~in both layers~~ are shown in figure 1~~.

For future reference, we note that ~~equations~~ (4.1) may be written ~~equivalently~~ as

$$\bar{Q}_1(r) = \Gamma_1 + \Delta_1 \mathcal{H}(r - R_1), \quad \bar{Q}_2(r) = \Gamma_2 + (\beta r - \Gamma_2)\mathcal{H}(r - R_2), \tag{4.4}$$

where $\mathcal{H}(\cdot)$ is the Heaviside function, which is defined to vanish at zero: $\mathcal{H}(0) = 0$. The gradient of the basic PV profile (4.4) is

$$\frac{d\bar{Q}_1}{dr} = \Delta_1 \delta(r - R_1), \quad \frac{d\bar{Q}_2}{dr} = \Delta_2 \delta(r - R_2) + \beta \mathcal{H}(r - R_2). \tag{4.5}$$

4.2. *Integral eigenvalue equations*

Define $s_i(r, \theta, t)$ to be the displacement of a particle from its initial reference location at $t = 0$ ~~at~~in ~~the~~layer i~~th layer~~. Since ~~the~~ PV is conserved as it moves, the change in ~~the~~ PV at the new particle location ~~for small s_i,~~ is, for small s_i,

$$q_i(r + s_i, \theta, t) = \bar{Q}_i(r, \theta) - \bar{Q}_i(r + s_i, \theta) = -\frac{d\bar{Q}_i}{dr} s_i \tag{4.6}$$

(cf. Ref. **?**). If all ~~the~~ displacements are associated with an azimuthal integer mode number m and frequency ω as in (3.3), then we may write $s_i = d_i(r)e^{i(m\theta - \omega t)}$ where $d_i(r)$ is the amplitude of the radial displacement of the particle. Comparing (4.6) with (3.3)~~, it is~~ shows clearly that

$$\mathcal{Q}_i(r) = -\frac{d\bar{Q}_i}{dr} d_i(r). \tag{4.7}$$

~~By~~Based on (4.5) and (4.7), $\mathcal{Q}_1$ vanishes everywhere except at $r = R_1$, where it is given by a delta function. The displacement s_1 of a particle at $r = R_1$ can also be interpreted as the deformation of C_1 (cf. Refs. **??**); the amplitude $d_1(R_1)$ of ~~C_1's~~the perturbation C_1 is denoted by α_1/R_1. Similarly, the amplitude $d_2(R_2)$ of ~~C_2's~~ perturbation C_2 at $r = R_2$ is denoted by α_2/R_2. The amplitude of the displacement $d_2(r)$ at the outer region $r > R_2$ in the lower layer is denoted by $\eta(r)/r$~~,~~ and can be ~~viewed~~interpreted as the deformation of the background constant-PV contours (which are circles). The division by R_1, R_2, and r is ~~made~~done ~~in order~~ to make the integral operator symmetric, which is useful as is shown below (§6). Therefore, using (4.5) and (4.7), we write

$$\mathcal{Q}_1 = -\frac{\Delta_1 \alpha_1}{R_1}\delta(r - R_1), \quad \mathcal{Q}_2 = -\frac{\Delta_2 \alpha_2}{R_2}\delta(r - R_2) - \frac{\beta}{r}\eta(r)\mathcal{H}(r - R_2). \tag{4.8}$$

Inserting (4.8) into the eigenvalue integral equations (3.19) yields the following three eigenvalue equations~~;~~:

$$\frac{\bar{V}_1(R_1) - \Delta_1 G_{11}(R_1, R_1)}{R_1}\alpha_1 - \frac{\Delta_1 G_{12}(R_1, R_2)}{R_2}\alpha_2 - \beta\Delta_1 \int_{R_2}^{\infty} \frac{G_{12}(R_1, r')}{r'}\eta(r')dr' = \frac{\omega}{m}\alpha_1, \tag{4.9}$$

$$-\frac{\Delta_2 G_{21}(R_2, R_1)}{R_1}\alpha_1 + \frac{\bar{V}_2(R_2) - \Delta_2 G_{22}(R_2, R_2)}{R_2}\alpha_2 - \beta\Delta_2 \int_{R_2}^{\infty} \frac{G_{22}(R_2, r')}{r'}\eta(r')dr' = \frac{\omega}{m}\alpha_2, \tag{4.10}$$

$$-\frac{G_{21}(r, R_1)}{R_1}\alpha_1 - \frac{G_{22}(r, R_2)}{R_2}\alpha_2 + \frac{\bar{V}_2(r)}{r}\eta(r) - \beta \int_{R_2}^{\infty} \frac{G_{22}(r, r')}{r'}\eta(r')dr' = \frac{\omega}{m}\eta(r). \tag{4.11}$$

Equations (4.9)–(4.11) can be recast into a standard matrix eigenvalue equation~~; it is~~, which can then be solved numerically by using the "eig" function in Matlab, which uses the QZ algorithm (**?**). To get the matrix form, the integrals are approximated via a

Gaussian quadrature rule (see, e.g., Ref. ?), which for any function $f(x)$ takes the form $\int_{R_2}^{\infty} f(r)dr \approx \sum_{i=1}^{N} w_i f(r_i)$; here r_i and w_i are the nodes and weights, respectively, of the quadrature rule employed.

Since the domain is infinite, we divide~~d~~ the integral into two regions: The first ~~one~~region is the ~~close~~near neighborhood of the island, where the basic flow velocities at the two layers are significant. Outside the largest ring, the velocities drop exponentially with r with typical length scale $\Lambda/\sqrt{\lambda_1\lambda_2}$ (see Appendix A); thus the velocities remain significant at $R_2 \leq r \leq \max(R_1, R_2) + 5\Lambda/\sqrt{\lambda_1\lambda_2}$. In this region, the Legendre-Gauss ~~Q~~quadrature rule is applied with 1000 points. The second region is outside, at $\max(R_1, R_2) + 5\Lambda/\sqrt{\lambda_1\lambda_2} < r < \infty$, where the Gauss-Laguerre quadrature rule is applied with 150 points. Tests of convergence show that the results are robust; e.g., even with half the number of points in each region, the error in calculating the eigenvalues~~,~~ is less than 0.1%.

5. Resonance viewpoint

The integral eigenvalue equations (4.9)–(4.11) allow direct interpretation of the couplings that occur in the system studied. We demonstrate it by using the first equation, (4.9), which determines the angular velocity of the upper contour C_1 perturbation, at $r = R_1$. The right-hand side (RHS) of (4.9) may also be viewed ~~also~~ as the time derivative of the PV perturbation at $r = R_1$ ~~since~~because the time derivative is proportional to ω ~~(~~[cf. ~~equation~~ (3.3)]. The first term ~~i~~on the left-hand side (LHS) of (4.9) contains the free-streaming term $\bar{V}_1(R_1)\alpha_1/R_1$ with ~~the~~ coupling to the basic PV jump at its place, namely, $-\Delta_1 G_{11}(R_1, R_1)\alpha_1/R_1$. This term would determine the angular velocity of the PV contour at $r = R_1$ ~~if~~were no other couplings to occur (cf. Ref. ?). The next term represents the coupling between the PV perturbations at C_1 and C_2~~, since it is~~because α_2 ~~t~~is what influences the time ~~development~~dependence of α_1. Finally, the integral term ~~represents the influence of~~dictates how the PV perturbation $\eta(r)/r$ in the lower layer outside C_2 ~~on~~affects the evolution of ~~C_1's~~ the perturbation at C_1.

~~The identification of e~~Each of the coupling terms can now b~~y~~e ~~applied in case that~~identified when instability is reached. By allowing only certain couplings to remain in the equations while removing others, ~~one can isolate~~ different subsystems of the entire system can be isolated and ~~find~~ the dominant ones found. These are the couplings that lead to the closest phase velocity and growth rate**[AU: "Closest" to what? Or do you mean "…couplings that lead to a phase velocity that is closest to the growth rate …"?]**of the fully coupled system. In this case, the PV perturbations that couple ~~to~~and thereby cause the instability are said to be resonant.

The resonance viewpoint has been ~~employed~~used by many authors for shallow-water systems, as mentioned in the Introduction. Usually, ~~it~~resonance is demonstrated by the ~~crossing~~intersection of two dispersion curves (?); ~~here another method is employed~~ in the QG case, ~~i.e. the method of~~we find~~ing~~ the dominant couplings in the eigenvalue equations written in terms of the PVs. ~~In~~ §7 ~~it is~~ show~~n~~s that the results are consistent with those ~~of~~obtained based on the ~~crossing~~intersection of two dispersion curve~~ method~~s.

In the resonan~~t~~ce viewpoint, the instability is caused by the interaction ~~of~~between two waves ~~which~~that phase-lock and thereby enhance ~~the~~each other's growth ~~of each other~~ (see, e.g., Refs. ??). For the basic flow considered ~~in this paper~~herein, ~~there are~~ three Rossby waves ~~which~~ can interact: the first ~~one~~wave travels along C_1 (where the PV in the upper layer jumps), the second ~~one~~wave travels along C_2, and the third ~~one~~wave, which exists ~~due to~~because of the bottom topography, travels ~~at~~in the outer region ~~in the second layer~~ ($r > R_2$) of the second layer. As is discussed in §6, ~~there are~~ various types of perturbations ~~types~~ exist for this third wave, which we collectively call "topographic"

Figure 2: Growth rates Im(ω) for different couplings as functions of R_2 at $R_1 = 5$, $\beta = -0.1$, $m = 2$, $\Lambda = 1$, and $\lambda_1 = \lambda_2 = 0.5$ for (a) $\Gamma_1 = 1$, (b) $\Gamma_1 = -1$. The types of couplings are as follows: Full (red), CC (blue), C1T (purple), C2T (brown), CCT (green).

perturbations. ~~U~~By using the eigenvalue equations (4.9)–(4.11), we identify four types of instabilit~~y~~ies:

(*a*) Contour-contour (CC) instability: ~~i~~In this case, the dominant interaction ~~which~~ lead~~s~~ing to instability is ~~the interaction~~that between ~~the~~ perturbations at the ring~~s'~~ periphery (i.e., between C_1 and C_2). The coupling ~~the~~ contours C_1 and C_2 alone correspond~~s~~ to setting $\beta = 0$ in (4.9)–(4.11), thus ~~remaining~~leaving ~~with~~ two algebraic equations to be solved. ~~In~~ §7 presents this instability ~~is presented~~ in more detail.

(*b*) Contour-C_1–topography (C_1T) instability: In this case the CC subsystem (composed of C_1 and C_2 alone) is stable~~, i.e.~~; that is, the PV-jumps alone ~~are~~do not ~~the~~ cause ~~t~~the ~~o~~ instability, but rather the resonance of the wave at C_1 with the topographic PV perturbations in the lower layer (in the region $r > R_2$). The eigenvalue calculation in this case is ~~achieved~~done by setting $\alpha_2 = 0$ in ~~equations~~ (4.9)–(4.11).

(*c*) Contour-C_2–topography (C_2T) instability: In this case, the wave ~~at~~on the lower layer PV contour is in resona~~tes~~nce with the ~~PV~~ topographic PV perturbations outside the contour. The eigenvalue calculation in this case is ~~achieved~~done by setting $\alpha_1 = 0$ in ~~equations~~ (4.9)–(4.11).

(*d*) Both-contours–topography (CCT) instability: In this case, the dominant resonance is between one of the neutral perturbation types of the mutual contours subsystem CC and the topographic perturbations. The eigenvalue calculation in this case is ~~achieved~~done by rearranging ~~the equations~~ (4.9)–(4.11) ~~in~~ such ~~a way~~ that the perturbations of the CC subsystem are decoupled; this is explained in §7.2. ~~We n~~Note that, although it seems that the entire system takes part in this instability, this is not so: only one of the neutral CC perturbation types participates in this resonance, ~~while~~whereas the other one ~~is~~does not.

In the following, we collectively ~~term~~call instabilities (ii)–(iv)**[AU: Do you mean (b)–(d)?]**~~as~~ contour~~s~~-topography (CT) instabilities; these are discussed in §7 in more detail. Figure 2 presents an example showing the identification of the types of instabilit~~y typ~~ies; the growth rates [i.e., Im(ω)] are shown for each of the above resonances~~,~~ as a function of the radius of the lower-layer ring. The flow parameters are $\beta = -0.1$, $R_1 = 5$, $\lambda_1 = \lambda_2 = 1/2$, and $\Gamma_1 = 1$ (Figure 2a) or $\Gamma_1 = -1$ (Figure 2b).

~~In the case of~~For $\Gamma_1 = 1$ (Figure 2a), the CC resonance ~~is~~ domina~~nt~~tes when $2.3 < R_2 < 5.5$, ~~while~~whereas the C_2T resonance ~~is~~ domina~~nt~~tes when $R_2 < 2$. The C_1T resonance is ~~totally~~completely absent in this case~~; the~~, as explai~~nation is give~~ned below (§5.1). The growth rates of the CC instability ~~are~~ generally ~~higher than~~exceed those of the C_2T instability. Also, there is a small "window" at $2 < r < 2.3$ where the CC interaction is stable ~~while~~whereas the C_2T interaction is not; ~~yet~~however, the growth rate of the full instability is much ~~higher~~greater (up to ~~4 times~~fourfold) than the growth rate of the C_2T resonance and therefore cannot be attributed to this resonance. The dominant instability in this region is of type CCT.

~~In the case of~~For $\Gamma = -1$ (~~f~~Figure 2b), again the CC resonance ~~admits~~leads to the ~~highest~~greatest growth rates and is dominant ~~in~~for most values of R_2. In much of the CC instability region, the actual (full-system) growth rate is ~~lower~~less than that implied by the growth rate of the CC interaction. Therefore, the topography in this case stabilizes the flow. Again at small values of R_2 (below 1.5), the dominant resonance is between one of the contours and the topographic perturbations, but this time it is of type C_1T, and the C_2T type is absent. At small regions ~~at~~ $1.5 < r < 2.1$, ~~again~~ the instability is again of type CCT.

5.1. *Pseudomomentum considerations*

~~As is known,~~Although ~~while~~ momentum is not a conserved quantity in the system (3.2) of ~~the~~ linearized equations ~~(3.2)~~, one can define an analogous quantity that is conserved, namely, the pseudomomentum (**?**). ~~While~~Whereas a necessary condition for instability to occur is phase~~-~~-locking (i.e., ~~crossing~~the intersection of the dispersion curves of two neutral waves), not every ~~crossing~~intersection leads to instability. As shown ~~by~~in Ref. **?**, an additional requirement for instability is that the two waves ~~would~~ have pseudomomenta of opposite signs ~~of pseudomomentum~~.

The expression for the pseudomomentum density in the two~~-~~-layer model on the beta cone (i.e., in polar coordinates where the basic flow is radially symmetric) is developed in Appendix C and is given by ~~the following expression,~~

$$\mathcal{M} = -\frac{\lambda_1}{2}\frac{d\bar{Q}_1}{dr}\left\langle s_1^2\right\rangle - \frac{\lambda_2}{2}\frac{d\bar{Q}_2}{dr}\left\langle s_2^2\right\rangle, \tag{5.1}$$

where the brackets $\langle\cdot\rangle$ denote the azimuthal average of the variable. The pseudomomentum density satisfies the continuity equation

$$\frac{\partial\mathcal{M}}{\partial t} + \frac{1}{r}\frac{\partial\mathcal{F}}{\partial r} = 0, \tag{5.2}$$

where $\mathcal{F} = \langle\frac{\partial\psi_1}{\partial\theta}\frac{\partial\psi_1}{\partial r}\rangle$ is the Eliaseen-Palm flux. If (5.2) is integrated over the entire plane outside the island ($r > R$), we get the equation for pseudomomentum conservation, $\frac{\partial M}{\partial t} = 0$, where

$$M = \int_R^\infty r\mathcal{M}dr = -\int_R^\infty\left(\frac{\lambda_1}{2}\frac{d\bar{Q}_1}{dr}\left\langle s_1^2\right\rangle + \frac{\lambda_2}{2}\frac{d\bar{Q}_2}{dr}\left\langle s_2^2\right\rangle\right)dr. \tag{5.3}$$

Since M is conserved, it must vanish in the case of an instability ~~it must vanish~~, ~~This~~which leads to ~~the known~~ Rayleigh's necessary condition for instability~~,~~ that the basic PV gradient must be somewhere negative and somewhere positive (cf. **??**). Moreover, ~~in case that~~if only two perturbation types are in resona~~te~~nce, their pseudomomenta must have opposite signs (**?**).

For the basic flow considered ~~in this paper~~herein, the use of (4.5) and (4.8) gives a ~~the~~ pseudomomentum ~~becomes~~of

$$M = \left(-\frac{\lambda_1\Delta_1}{4R_1^2}|\alpha_1|^2 - \frac{\lambda_2\Delta_2}{4R_2^2}|\alpha_2|^2 - \int_R^\infty\frac{\lambda_2\beta}{4r^2}|\eta|^2dr\right)e^{2\mathrm{Im}(\omega)t}. \tag{5.4}$$

Since the PV jumps ~~at~~when the two contours C_1 and C_2 ~~are~~have opposite ~~in~~ signs, their pseudomomenta ~~are~~have opposite ~~in~~ signs and the Rossby waves traveling along these contours may be in resonance. The pseudomomentum of the perturbation at the exterior region $r > R_2$ is always positive, β being always negative. Therefore, the exterior perturbations can only be in resona~~te~~nce with the contour wave whose pseudomomentum is negative~~,~~ (i.e., ~~is~~ traveling along a positive PV gradient).

This explains why only one contour wave resonates with the topographic perturbations, as shown in ~~f~~Figure 2. If $\Gamma_1 = 1$, the pseudomomentum of the contour wave at $r = R_1$ is positive ~~(since~~[because $\Delta_1 < 0$ by (4.2)~~))~~], ~~while~~whereas ~~that~~the pseudomomentum of the contour wave at $r = R_2$ is negative ~~(since~~[because $\Delta_2 > 0$ for the specific parameters ~~chosen~~dictated by (4.2)~~)~~]. Thus, only the lower-layer contour has ~~opposite sign of~~

pseudomomentum opposite in sign relative to ~~that of~~ the outside perturbations (which ~~is~~ always have positive pseudomomentum)~~;~~, ~~therefore~~so a C_1T instability is impossible in this case (~~f~~Figure 2a). The same argument explains why a C_2T instability is impossible ~~in the case~~for $\Gamma_1 = -1$ (~~f~~Figure 2b).

6. Perturbation types in outer region

We now focus on the subsystem of the basic flow outside the liquid contours, ~~i.e.~~which means that we search for modes whose perturbation is dominant (i.e., strong relative to the contours' perturbations) ~~at~~in the lower layer at $r > R_2$. For this, we assume that the PV-jumps at any of the liquid contours ~~is~~are negligible ($\Delta_1 \approx \Delta_2 \approx 0$), thus avoiding any coupling to waves at th~~at~~e given contour~~s~~. The resulting PV perturbations can be seen as self-excitations of the outer region ~~,~~ caused by ~~the presence of~~ the topography. ~~Physically, a~~As a consequence, the contours may physically oscillate and resonate to yield the ~~contours-topography~~CT instability, which is ~~which is~~ discussed in §7.

Neglecting α_1 and α_2 in (4.9)–(4.11) yields a single integral equation~~s~~ for η,

$$\frac{\bar{V}_2(r)}{r}\eta(r) - \beta\int_{R_2}^{\infty}\frac{G_{22}(r,r')}{r'}\eta(r')dr' = \frac{\omega}{m}\eta(r). \tag{6.1}$$

Since the kernel $\frac{G_{22}(r,r')}{r'}$ is symmetric (see Appendix B), the operator on the ~~left hand side~~LHS of the equation~~,~~ acting on $\eta(r)$~~,~~ is symmetric; ~~therefore~~so the eigenvalues are necessarily real. The eigenfunctions are orthogonal with respect to the standard inner product defined by $\langle f_1, f_2\rangle = \int_{R_2}^{\infty} f_1(r)f_2^*(r)dr$ for any two functions f_1 and f_2, for which this integral is convergent. This integral equation (6.1) is similar in form to the integral equation of ~~a~~the ~~barotropic~~BT flow discussed in detail by **?**. Here the Green's function ~~is~~ differ~~ent~~s ~~due to~~because of the cylindrical symmetry and, more importantly, ~~due to~~because of the ~~baroclinic~~BC component of the Green's function [see (3.18)]. Another difference is the fact that the domain here is unbounded.

Equation (6.1) was solved numerically by using the numerical scheme described at the end of §4.2. The frequency ω ~~was~~is indeed ~~found to be~~ always real, and Figure 3 shows some eigenfunction~~s~~ examples ~~found are shown in figure 3~~. The properties of the spectrum and ~~the~~ eigenfunctions are explained analytically below.

Before ~~dwelling~~discussing ~~into~~ the structure of the solutions ~~structure~~, we make a rough estimate of the allowed frequencies (the spectrum) ~~may be carried out~~. Multiplication of ~~equation~~ (6.1) by η^* and integrating yields

$$\int_{R_2}^{\infty}\frac{\bar{V}_2(r)}{r}|\eta(r)|^2dr - \beta\int_{R_2}^{\infty}\int_{R_2}^{\infty}\frac{G_{22}(r,r')}{r'}\eta^*(r')\eta(r)dr'dr = \frac{\omega}{m}\int_{R_2}^{\infty}|\eta(r)|^2dr. \tag{6.2}$$

Since the term $\frac{G_{22}(r,r')}{r'}$ is always negative [by (3.18), (B 6), and (B 12)~~)~~], the second integral ~~i~~on the LHS is negative~~; therefore~~, so the possible values of ω are

$$-\infty < \omega < \sup_r \frac{m\bar{V}_2(r)}{r}, \tag{6.3}$$

where "sup" denotes the supremum. For future reference, we define the segments as

$$\mathcal{S}_1 = \left(\inf_r \frac{m\bar{V}_2(r)}{r}, \sup_r \frac{m\bar{V}_2(r)}{r}\right), \quad \mathcal{S}_2 = \left(-\infty, \inf_r \frac{m\bar{V}_2(r)}{r}\right), \tag{6.4}$$

so by (6.3), $\omega \in \mathcal{S}_1 \cup \mathcal{S}_2$. We note that, contrary to the ~~derived~~ bounds ~~on~~derived for the phase velocity given by **?** for annular flows (known as the semi-circle theorems, cf.

Figure 3: Examples of perturbation types ~~at~~in the outer region $r > R_2$. The shared flow parameters are $\beta = -0.1$, $\Lambda = 1$, $\lambda_1 = \lambda_2 = 0.5$, $m = 3$. ~~A~~The arrow designates a delta function. (a) ~~a~~Asymptotically wavelike ~~barotropic~~BT mode with critical layer ($R_1 = 4$, $R_2 = 5$, $\Gamma_1 = 1$), (b) ~~a~~Asymptotically wavelike ~~barotropic~~BT mode without critical layer ($R_1 = 2$, $R_2 = 3.5$, $\Gamma_1 = -1$), (c) ~~a~~Asymptotically evanescent ~~barotropic~~BT mode with critical layer ($R_1 = 2$, $R_2 = 3.5$, $\Gamma_1 = -1$), (d) ~~a~~Asymptotically ~~baroclinic~~BC mode without critical layer ($R_1 = 2$, $R_2 = 3.5$, $\Gamma_1 = -1$). Arrows ~~denote~~designate delta functions, and their height corresponds to the ~~multiplicative~~ prefactor ~~in~~multiplying ~~front of~~ the delta functions.

?), the phase velocity here cannot, by similar arguments, be bounded from below~~. The reason is that~~ because the flow is unbounded, whereas such theorems use the fact that ~~it~~the flow is confined to a channel (zonal or annular).

6.1. *Structure of solution near a critical layer*

When $\omega \in \mathcal{S}_1$ there is a critical distance r_c at which the angular velocity ω/m of the perturbation is equal to the angular velocity $V(r_c)/r_c$ of the flow; the integral equation (6.1) is then singular. In this case, the solution contains a critical layer (see, e.g., Ref. ?). The ~~left hand side~~LHS of the equation can be viewed as ~~a~~the sum of an operator of multiplication by $\bar{V}_2/r$ and an integral operator. Following Refs. ??, the solution is written in the form of a delta function (the eigenfunction of the multiplication operator) plus an additional term,

$$\eta(r) = D(\omega)\delta\left(\frac{\bar{V}_2}{r} - \frac{\omega}{m}\right) - P\frac{\beta}{\frac{\bar{V}_2}{r} - \frac{\omega}{m}}\xi(r), \tag{6.5}$$

where $D(\omega)$ and $\xi(r)$ are unknown functions to be specified, $\xi(r)$ is assumed to be a regular function of r, P signifies that the principal value of the integral is to be taken when integrating the last expression with respect to r [i.e., $P\int_R^\infty = \lim_{\epsilon\to 0}(\int_R^{r_c-\epsilon} + \int_{r_c+\epsilon}^\infty)$]. Some of the solutions obtained numerically are indeed of the form (6.5), as shown in Figures 3a and 3c; the PV perturbation blows up near the point $r = r_c$ and a delta function appears at $r = r_c$. Note that equation (6.5) is valid also if there is no r for which $V(r)/r = \omega/m$, since then there is no critical layer and we may set $D(\omega) = 0$. These regular solutions are shown in Figures 3b and 3d.

Plugging (6.5) into (6.1) yields the following equation for ξ:

$$\xi(r) = -\frac{D(\omega)G_{22}(r, r_c)}{|(V(r)/r)'_{r_c}|r_c} + P\int_{R_2}^\infty \frac{\beta G_{22}(r, r')}{\left(\frac{\bar{V}_2(r')}{r'} - \frac{\omega}{m}\right) r'}\xi(r')dr', \tag{6.6}$$

where we assume for simplicity that, ~~at~~for $r > R_2$, the function V_2/r is injective and $r_c = (V_2/r)^{-1}(\omega/m)$, as is the case for the basic flows considered ~~in this paper~~herein. Also, we used the mathematical relation $\delta(f(x)) = \delta(x - x_0)/|f'(x_0)|$ that holds for any smooth injective function $f(x)$, where x_0 is a root of $f(x)$ ~~(~~[if x_0 exists, otherwise $\delta(f(x)) = 0$~~)~~]. Since (6.1) is homogeneous, we can arbitrarily demand that

$$\int_{R_2}^\infty \eta(r')dr' = 1, \tag{6.7}$$

which, by (6.5), is equivalent to the specification of the function $D(\omega)$ by the following equation:

$$\frac{D(\omega)}{|(V(r)/r)'_{r_c}|} - P\int_{R_2}^{\infty} \frac{\beta}{\frac{\bar{V}_2(r')}{r'} - \frac{\omega}{m}} \xi(r')dr' = 1. \tag{6.8}$$

Using (6.8) in (6.6), we get that ξ satisfies the following nonsingular inhomogeneous Fredholm equation of the second kind:

$$\xi(r) = -\frac{G_{22}(r, r_c)}{r_c} + \beta\int_{R_2}^{\infty} \frac{G_{22}(r, r')/r' - G_{22}(r, r_c)/r_c}{\frac{\bar{V}_2(r')}{r'} - \frac{\omega}{m}} \xi(r')dr'. \tag{6.9}$$

The nonsingularity is guaranteed since the derivative of the Green's functions~~' derivative~~ is always much ~~lower~~less than the derivative of the velocity~~'s derivative~~ (inverse power ~~vs~~versus~~.~~ linear function of r; see Appendixes A and B). Since there is no singularity in this equation at $r = r_c$, the function $\xi(r)$ is regular, as assumed. If ω is outside the range of $\{mV(r)/r\}$, then the solution consists only of the regular function $\xi(r)$ with no blowup.

Equation (6.9) can be transformed to a fourth-order nonhomogeneous (homogeneous) differential equation if a critical layer exists (does not exist~~s~~) by the procedure presented in Appendix D. The nonhomogeneous term (D 6) that appears in the resulting differential equation (D 5) contains only a delta function with its derivatives, ~~that~~which are singular only at $r = r_c$ (if they exist~~s~~). Therefore, ~~asymptotically~~ at $r \to \infty$ the solutions to the differential equation ~~have~~asymptotically approach the same form, regardless of whether ~~there is~~ a critical layer ~~or not~~is present. We denote the four linearly independent regular solutions to the equation by h_1, h_2, h_3, and h_4. In the following section we find asymptotic expressions for ~~h_j $(j = 1..4)$~~h_j $(j = 1, \ldots, 4)$ and find the spectrum properties of the eigenvalue equation (6.1).

6.2. *Asymptotically barotropic and baroclinic wave types*

We now show that there are two types of solution that, asymptotically at large r, behave as ~~barotropic~~BT and ~~baroclinic~~BC waves. For this we resort to the equations in their differential form~~, equation~~ (3.4)~~,~~ and consider solutions far from the origin, where $\bar{V}_2$ can be neglected~~;~~. ~~since~~Because the velocity diminishes exponentially with r [see ~~equations~~ (A 20) and (A 21)], such a range may always be found. By (3.4), far from the origin, $\mathcal{Q}_1 = 0$ and $\mathcal{Q}_2 = -m\beta\Phi_2/\omega r$, so equations (3.5) and (2) become

$$0 = \frac{d^2\Phi_1}{dr^2} + \frac{1}{r}\frac{d\Phi_1}{dr} - \frac{m^2}{r^2}\Phi_1 - \frac{\Lambda^2}{\lambda_1}(\Phi_1 - \Phi_2), \tag{6.10}$$

$$-\frac{m\beta\Phi_2}{\omega r} = \frac{d^2\Phi_2}{dr^2} + \frac{1}{r}\frac{d\Phi_2}{dr} - \frac{m^2}{r^2}\Phi_2 + \frac{\Lambda^2}{\lambda_2}(\Phi_1 - \Phi_2). \tag{6.11}$$

Asymptotically we may neglect the ~~left hand side~~LHS of (6.11) since Φ_2 appears ~~i~~on the ~~right hand side~~RHS without division by r. We use the ansatz $\Phi_2 = a\Phi_1$, where a is some parameter to be determined. As ~~is~~ shown below, this ansatz leads to four independent solutions, which, by the ~~abovesaid~~discussion at the end of §6.1, cover all the possible asymptotic solutions of the fourth-order differential equation. Plugging $\Phi_2 = a\Phi_1$ into ~~equations~~ (6.10) and (6.12) ~~we get~~gives the set of equations (after dividing the second equation by a)~~,~~

$$0 = \frac{d^2\Phi_1}{dr^2} + \frac{1}{r}\frac{d\Phi_1}{dr} - \frac{m^2}{r^2}\Phi_1 - \frac{\Lambda^2}{\lambda_1}(1 - a)\Phi_1, \tag{6.12}$$

$$0 = \frac{d^2\Phi_1}{dr^2} + \frac{1}{r}\frac{d\Phi_1}{dr} - \frac{m^2}{r^2}\Phi_1 + \frac{\Lambda^2}{\lambda_2}\frac{(1 - a)}{a}\Phi_1. \tag{6.13}$$

These two equations are identical provided that

$$\frac{1-a}{\lambda_2 a} = -\frac{1-a}{\lambda_1} \Rightarrow a = 1 \text{ or } a = -\frac{\lambda_1}{\lambda_2}. \tag{6.14}$$

Thus, asymptotically, $\Phi_2 \sim \Phi_1$ or $\Phi_2 \sim -\frac{\lambda_1}{\lambda_2}\Phi_1$. The first expression corresponds to the asymptotically ~~barotropic~~BT mode, where $\Phi_{\rm BC} = \Phi_1 - \Phi_2 \approx 0$, and the second expression corresponds to the asymptotically ~~baroclinic~~BC mode, where $\Phi_{\rm BT} = \lambda_1\Phi_1 + \lambda_2\Phi_2 \approx 0$. In the following, we loosely call perturbations whose asymptotic behavior is ~~barotropic~~BT (~~baroclinic~~BC) as ~~barotropic~~BT (~~baroclinic~~BC) modes, without repeating the fact that this behavior is only asymptotic. Also, note that the ~~barotropic~~BT or ~~baroclinic~~BC character of the mode is reflected only in the streamfunctions and not in the relations between the PV perturbations, because the PV perturbation at the upper layer is zero in any case; thus, the ~~barotropic~~BT and ~~baroclinic~~BC PV perturbations are $\mathcal{Q}_{\rm BT} = \lambda_2\mathcal{Q}_2$ and $\mathcal{Q}_{\rm BC} = -\mathcal{Q}_2$ (i.e., they are of the same order of magnitude). Having arrived to the conclusion that there are two kinds of asymptotic modes, we now turn to find their r dependence.

6.2.1. *Barotropic mode*

First, we assume that the ~~barotropic~~BT component of the streamfunction is the dominant ~~one~~component, $\Phi_{\rm BT} \gg \Phi_{\rm BC}$ (i.e., $\Phi_1 \approx \Phi_2$). By (3.11), this means that

$$\int_{R_2}^{\infty} \frac{G_{\rm BT}(r,r')}{r'}\eta(r')dr' \gg \int_{R_2}^{\infty} \frac{G_{\rm BC}(r,r')}{r'}\eta(r')dr', \tag{6.15}$$

so we take only the ~~barotropic~~BT component of the Green's function in (6.1),

$$\frac{\bar{V}_2(r)}{r}\eta(r) - \beta\int_{R_2}^{\infty} \frac{\lambda_2 G_{\rm BT}(r,r')}{r'}\eta(r')dr' = \frac{\omega}{m}\eta(r). \tag{6.16}$$

We ~~impose~~apply the linear operator D_1 defined by (D 1) on both sides of (6.16) and use (D 2). The term $\mathrm{D}_1(\frac{\bar{V}_2(r)}{r}\eta(r))$ is neglected because the basic velocity and its derivatives are negligible far from the island (see Appendix A). The integral equation is then converted to the differential equation

$$\frac{\eta(r)}{r} = -\frac{\omega}{m\beta\lambda_2}\left(\frac{d^2\eta}{dr^2} + \frac{1}{r}\frac{d\eta}{dr} - \frac{m^2}{r^2}\eta\right). \tag{6.17}$$

If $\omega < 0$, the general solution to (6.17) is given by

$$\eta(r) = AH_{2m}^{(1)}\left(2\sqrt{\frac{m\beta\lambda_2 r}{\omega}}\right) + BH_{2m}^{(2)}\left(2\sqrt{\frac{m\beta\lambda_2 r}{\omega}}\right), \tag{6.18}$$

where $H_{2m}^{(1)}$ and $H_{2m}^{(1)}$ are the Hankel functions of the first ~~kind~~ and second kinds, respectively, of order $2m$. ~~Referring to abovesaid~~Based on the discussion at the end of §6.2, we denote the two regular solutions to (D 5), ~~that~~for which $H_{2m}^{(1)}$ and $H_{2m}^{(2)}$ are their asymptotic approximation, by h_1 and h_2, respectively. The solutions must obey the radiation condition, according to which energy cannot arrive from outside; this no-radiation condition is satisfied only by $H_{2m}^{(1)}$, so $B = 0$ (see **?** for details). The solution to (6.16) is then in the form

$$\eta(r) = Ah_1(r;\omega) + D(\omega)\delta\left(\bar{V}_2/r - \omega/m\right), \tag{6.19}$$

where asymptotically $h_1(r;\omega) \sim H_{2m}^{(1)}(2\sqrt{m\beta\lambda_2 r/\omega})$ and $D(\omega)$ is nonzero if there is a critical layer~~,~~ and zero otherwise. Substitution of (6.19) into (6.16) and applying ~~the~~

~~equation at~~ $r = R_2$ leads to two options: (i) If $\omega \in \mathcal{S}_1$ [i.e., $D(\omega) \neq 0$], then A is nonzero and is determined by an inhomogeneous equation. Therefore, a solution exists for any $\omega \in \mathcal{S}_1$ ~~there is a solution~~. Such a solution, having a critical layer and asymptotically ~~barotropic~~BT, is shown in ~~f~~Figure 3a. (ii) ~~On the other hand~~Conversely, if ~~there is~~ no critical layer exists, then $D(\omega) = 0$, and the equation is homogeneous in A. Therefore, in this case, ω ~~can~~ takes only discrete values in ~~the~~ segment $\mathcal{S}_2$. Such an asymptotically ~~barotropic~~BT solution without a critical layer is shown in ~~f~~Figure 3b.

If $\omega > 0$, the general solution to (6.17) is given by a superposition of the modified Bessel functions of order $2m$,

$$\eta(r) = \tilde{A} K_{2m}\left(2\sqrt{-\frac{m\beta\lambda_2 r}{\omega}}\right) + \tilde{B} I_{2m}\left(2\sqrt{-\frac{m\beta\lambda_2 r}{\omega}}\right). \tag{6.20}$$

These two functions are the asymptotic approximations to h_1 and h_2 in this case. For the solutions to be limited as $r \to \infty$, we must set $\tilde{B} = 0$. In virtue of (6.3), the case $\omega > 0$ occurs only if $\bar{V}_2$ is positive, in which case it ~~is~~ approach~~ing 0~~es zero at infinity (see Appendix A). Therefore, $\omega \in \mathcal{S}_1$ and this type of perturbation always contains a critical layer:

$$\eta(r) = A h_1(r;\omega) + D(\omega)\delta\left(\bar{V}_2/r - \omega/m\right), \tag{6.21}$$

with $D(\omega) \neq 0$. An example of this solution is shown in ~~f~~Figure 3c. Substitution of (6.21) into (6.16) and ~~applying the equation at~~using $r = R_2$ leads to ~~a determination of~~ $A \neq 0$ with no limitation on ω. Therefore, ω can take any value in the segment $\mathcal{S}_1$.

6.2.2. *Baroclinic mode*

~~Now w~~We now ~~turn to~~consider the case where the ~~baroclinic~~BC component ~~is~~ domina~~nt~~tes. In this case, by (6.1),

$$\int_{R_2}^{\infty} \frac{\lambda_1 G_{\mathrm{BC}}(r,r')}{r'}\eta(r')dr' = -\frac{\omega}{m\beta}\eta(r). \tag{6.22}$$

We ~~impose~~apply the linear operator D_2 defined by (D 1) on both sides of (6.22) and ~~get~~, by using (D 2), obtain

$$\frac{\eta(r)}{r} = -\frac{\omega}{m\beta\lambda_1}\left(\frac{d^2\eta}{dr^2} + \frac{1}{r}\frac{d\eta}{dr} - \frac{m^2}{r^2}\eta - \frac{\Lambda^2}{\lambda_1\lambda_2}\eta\right). \tag{6.23}$$

The general solution to (6.23) is

$$\eta(r) = \frac{E}{\sqrt{r}} W_{\kappa,m}\left(\frac{2\Lambda r}{\sqrt{\lambda_1\lambda_2}}\right) + \frac{F}{\sqrt{r}} M_{\kappa,m}\left(\frac{2\Lambda r}{\sqrt{\lambda_1\lambda_2}}\right), \tag{6.24}$$

where $\kappa = \frac{m\beta\lambda_1\sqrt{\lambda_1\lambda_2}}{2\Lambda\omega}$, ~~and~~ $W_{\kappa,m}$ and $M_{\kappa,m}$ are the Whittaker functions of order (κ, m)~~;~~, and E and F are constants. ~~U~~By using the asymptotic form of the Whittaker functions~~'~~ ~~asymptotic form~~ (?), ~~asymptotically~~ we get asymptotically

$$\eta(r) \sim E r^{\kappa-\frac{1}{2}} e^{-\frac{\Lambda r}{\lambda_1\lambda_2}} + F r^{-\kappa-\frac{1}{2}} \frac{\Gamma(1+2m)}{\Gamma(\frac{1}{2}+m-\kappa)} e^{\frac{\Lambda r}{\lambda_1\lambda_2}}. \tag{6.25}$$

For the solution to be limited as $r \to \infty$, we set $F = 0$. ~~It is known~~Note that $W_{k,m}$ has positive zeros only if $\kappa > 1/2$ (?). This inequality yields $m\beta\sqrt{\lambda_1\lambda_2}/\Lambda \leq \omega < 0$, i.e., only negative eigenvalues yield ~~a~~ wavelike asymptotic eigenfunctions, an example of which is shown in ~~f~~Figure 3d.

Referring now to ~~abovesaid~~the discussion at the end of §6.2, we denote by h_3 and

~~s~~Segment containing ω	Sign of ω ~~sign~~	~~a~~Asymptotic mode	~~a~~Asymptotic form	~~s~~Spectrum is continuous~~/~~ or discrete
$\mathcal{S}_1$	$\omega > 0$	BT	$CK_{2m}(2\sqrt{m\beta\lambda_2 r/\omega})$	continuous
$\mathcal{S}_1$	$\omega < 0$	BT	$AH^{(1)}_{2m}(2\sqrt{m\beta\lambda_2 r/\omega})$	continuous
$\mathcal{S}_1$	any $\omega \neq 0$	BC	$EW_{\kappa,m}(2\Lambda r/\sqrt{\lambda_1\lambda_2})/\sqrt{r}$	continuous
$\mathcal{S}_2$	$\omega < 0$	BT	$AH^{(1)}_{2m}(2\sqrt{m\beta\lambda_2 r/\omega})$	discrete
$\mathcal{S}_2$	any $\omega \neq 0$	BC	$EW_{\kappa,m}(2\Lambda r/\sqrt{\lambda_1\lambda_2})/\sqrt{r}$	discrete

Table 1: Parts of the spectrum and ~~their~~the associated properties. The segments $\mathcal{S}_1$ and $\mathcal{S}_2$ are defined by (6.4). ~~*BT*~~BT and ~~*BC*~~BC designate ~~barotropic~~BT mode and ~~baroclinic~~BC mode, respectively.

h_4, respectively, the two regular solutions to (D 5), ~~that~~with $W_{\kappa,m}\left(2\Lambda r/\sqrt{\lambda_1\lambda_2}\right)/\sqrt{r}$ and $M_{\kappa,m}\left(2\Lambda r/\sqrt{\lambda_1\lambda_2}\right)/\sqrt{r}$ ~~are~~being their asymptotic approximation~~by h_3 and h_4, respectively~~. The general solution to (6.22) is then ~~in~~of the form

$$\eta(r) = Eh_3(r;\omega) + D(\omega)\delta(\bar{V}_2/r - \omega/m), \tag{6.26}$$

where asymptotically $h_3(r;\omega) \sim H^{(1)}_{2m}(2\sqrt{m\beta\lambda_2 r/\omega})$ and ~~where~~ $D(\omega)$ is nonzero if there is a critical layer and zero otherwise. Substitution of (6.26) into (6.22) and applying ~~the equation at~~ $r = R_2$ leads, as in the ~~barotropic~~BT case, to ~~to~~ two options: (i) If $\omega \in \mathcal{S}_1$ [i.e., $D(\omega) \neq 0$], then E is nonzero and is determined by an inhomogeneous equation. Therefore, a solution exists for any $\omega \in \mathcal{S}_1$~~ there is a solution~~. ~~On the other hand~~(ii) Conversely, if ~~there is~~ no critical layer exists, then $D(\omega) = 0$, and the equation is homogeneous in E. Therefore, in this case, ω can take only discrete values in the segment $\mathcal{S}_2$. Such an asymptotically ~~baroclinic~~BC solution without a critical layer is shown in ~~f~~Figure 3d. The solution is wavelike in some region and then, starting from some distance, decays exponentially in r, as implied by (6.25).

A summary of the different parts of the spectrum is listed in~~ t~~ Table 1. One part consists of all the values in ~~the~~ segment $\mathcal{S}_1$ (excluding zero), where each value has multiplicity 2 (i.e., there are two corresponding eigenfunctions with a critical layer). These eigenfunctions correspond asymptotically to ~~barotropic~~BT or ~~baroclinic~~BC forms. These are evanescent if $\omega > 0$. I~~n case that~~f $\omega < 0$, the asymptotically ~~barotropic~~BT type is wavelike as $r \to \infty$, and the asymptotically ~~baroclinic~~BC type is wavelike in a finite region if $\omega > m\beta\sqrt{\lambda_1\lambda_2}/\Lambda$ and ~~else~~otherwise is evanescent. ~~Other part~~The rest of the spectrum is a discrete set of the segment $(-\infty, \inf\{m\bar{V}_2/r\})$, including asymptotically ~~barotropic~~BT and ~~baroclinic~~BC types without a critical layer.

6.3. *Decay of asymptotically BT and BC modes*

The modal analysis above in §6.2 shows that the perturbation types belonging to the continuous spectrum are neutral (i.e., are maintained without growth or damping with time). However, ~~it is known that~~a correct treat~~ing~~ment of the initial-value problem correctly shows that such modes may give rise to asymptotic ~~alegbraic~~algebraic decay with time (?)~~,~~ or to algebraic grow~~ing~~th (see, e.g., Ref. ?). Since the perturbation expressions in the complex-ω plane contain poles and branch cuts, as ~~is~~ seen from (6.18) or (6.24), a natural question is what is their contribution to the flow stability properties?~~.~~ As ~~is~~

Figure 4: (a) Schematic drawing of the location on the complex-ω plane of the poles and branch lines of the Laplace transform response. There are poles due to the discrete spectrum [where $D(\omega)=0$], a pole at $\omega = m\bar{V}_2(r_0)/r_0$, a pole at $\omega = m\bar{V}_2(r)/r$, a pole at $\omega=0$ due to the asymptotically BC mode, a branch line at $\mathcal{S}_1$, and a branch line at $\mathrm{Im}(\omega)<0$ due to the asymptotically BT mode. Also, the Bromwich contour $\mathrm{Im}(\omega)=\gamma$ is designated. (b) ~~the c~~Contour for calculating the inverse Laplace transform of the asymptotically ~~barotropic~~BT mode.

shown in this section, these types contribute to its stability by causing decay~~ing~~ rather than neutrality of a given initial perturbation.

Consider a time-dependent PV perturbation of azimuthal mode number m in the lower layer, $q_2(r,\theta,t)=\zeta_2(r,t)e^{im\theta}$. Its Laplace transform is ~~defined as~~

$$\mathcal{Q}_2(r,\omega)=\int_0^{\infty}\zeta_2(r,t)e^{i\omega t}dt, \tag{6.27}$$

where the notation $\mathcal{Q}_i$ ~~is in agreement~~is consistent with the definition (3.3). The inverse Laplace transform is ~~given by~~

$$\zeta_2(r,t)=\frac{1}{2\pi}\int_{-\infty+i\gamma}^{\infty+i\gamma}\mathcal{Q}_2(r,\omega)e^{-i\omega t}d\omega, \tag{6.28}$$

where the Bromwich contour of integration is along $\mathrm{Im}(\omega)=\gamma$, where γ is greater than the imaginary part of all the singularities of $\mathcal{Q}_2(r,\omega)$.

Laplace-transforming the linearized equation for q_2 in (3.2) gives

$$\left(\frac{\bar{V}_2}{r}-\frac{\omega}{m}\right)\mathcal{Q}_2-\frac{\Phi_2}{r}\frac{d\bar{Q}_2}{dr}=\frac{\zeta_2(r,t=0)}{im}. \tag{6.29}$$

~~Let us a~~Assume for simplicity that $\zeta_2(r,t=0)=\delta(r-r_0)/r$ and denote the solution to (6.29) in this case by $\chi(r,r_0;\omega)/r$. This solution is the response function of the system to an initial delta-function perturbation. Thus, the equation for $\chi(r;r_0,\omega)$ is

$$\left(\frac{\bar{V}_2}{r}-\frac{\omega}{m}\right)\chi(r;r_0,\omega)-\beta\int_{R_2}^{\infty}\frac{G_{22}(r,r')}{r'}\chi(r';r_0,\omega)dr'=\delta(r-r_0). \tag{6.30}$$

Asymptotically, a~~t~~s $r\to\infty$ the solutions to (6.30) coincide with the solutions to (6.1), which by §6.2 ~~are~~ asymptotically ~~in~~take the form of Hankel functions of the first kind [see (6.18)] or Whittaker functions [see (6.24)]. Since, for large ω ?, ~~it is known that~~

$$H^{(1)}_{2m}\left(2\sqrt{\frac{m\beta r}{\omega}}\right)\sim\frac{1}{(2m)!}\left(\frac{m\beta r}{\omega}\right)^{2m},\quad W_{\kappa,m}\left(\frac{2\Lambda r}{\sqrt{\lambda_1\lambda_2}}\right)\sim r^{\frac{m\beta\lambda_1\sqrt{\lambda_1\lambda_2}}{2\Lambda\omega}-\frac{1}{2}}e^{-\frac{\Lambda r}{\lambda_1\lambda_2}}, \tag{6.31}$$

then χ is bounded as $|\omega|\to\infty$. It is therefore possible to deform the Bromwich contour integral until it consists only of integrals around poles and cuts.

~~In~~ Appendix E shows that the poles of $\chi(r,r_0;\omega)/r$ are ~~shown to be~~ of four types: (i) a discrete isolated set corresponding to perturbation types with no critical layer, (ii) the point $\omega=\bar{V}_2(r_0)/r_0$, (iii) a branch cut along ~~the~~ segment $\mathcal{S}_1$, and (iv) the poles of the regular functions $\xi(r;r_0,\omega)$ defined by (E 3). Here we use the asymptotic (~~at~~ $r\to\infty$) expressions for $\xi(r;r_0,\omega)$, which are identical to the asymptotic perturbations found in §6.2. Two types of singularities occur in the asymptotic regime: one is the singularity $1/\sqrt{\omega}$ that appears in (6.18), and the second is the singularity $1/\omega$ that appears in (6.24) (in the expression for κ). To account for the singularity $1/\sqrt{\omega}$, a branch of the square root must be chosen; for convenience we choose the branch cut to be on the negative imaginary axis. A schematic drawing illustrating the various poles and branch-cut locations is given in fFigure 4a. The singularity at $\omega = 0$ and the branch cut of $\sqrt{\omega}$ are unique to the beta-cone model.

The contribution of (i) to the inverse Laplace transform is a discrete sum of exponentials of the form $e^{-i\omega_n t}$, where $\{\omega_n\}$ is the discrete mentioned set mentioned. In the same way, the pole at $m\omega = \bar{V}_2(r_0)/r_0$ gives rise to a simple exponential $e^{-im\bar{V}_2(r_0)t/r_0}$. The contribution of the branch cut $\mathcal{S}_1$ results in algebraic decay as $1/t$ (**??**).

To calculate the contribution of the barotropicBT mode to the integral, we use the contour of integration used is as shown in fFigure 4b. The integral along the small circle, $\int_0^{2\pi} H_{2m}^{(1)}(2\sqrt{m\beta r/\epsilon e^{i\phi}})e^{-i\epsilon e^{i\phi}t}d\phi$, vanishes as $\epsilon \to 0$ (this can be found by direct numerical integration). Denoting the negative imaginary axis by the frequency by $\omega = ix$ alongthe negative imaginary axis, where x is real, the contribution to the integral (6.28) along the right side side of the branch cut is

$$\int_{-\infty}^{0} H_{2m}^{(1)}\left(2\sqrt{\frac{m\beta r}{ix}}\right)e^{-i\cdot ixt}dx = -\int_0^{\infty} H_{2m}^{(1)}\left(2\sqrt{\frac{m\beta r}{-ix}}\right)e^{-xt}dx. \tag{6.32}$$

Asymptotically as $t \to \infty$, significant contributions to the integral will come only from points x near zero. Therefore, the Hankel function in the integrand can be replaced by its asymptotic approximation at $x \sim 0$, which is

$$H_{2m}^{(1)}\left(2\sqrt{\frac{m\beta r}{-ix}}\right) \sim \left(\frac{-i}{m\pi^2\beta r}\right)^{1/4} x^{1/4}\exp\left(2i\sqrt{\frac{m\beta r}{-ix}} - im\pi - \frac{i\pi}{4}\right). \tag{6.33}$$

(**?**). Since the exponential term is bounded by 11, the integral in (6.32) is bounded by the following integral;:

$$\left(\frac{1}{m\pi\beta r}\right)^{1/4}\int_0^{\infty} x^{1/4}e^{-xt}dx = \left(\frac{1}{m\pi\beta r}\right)^{1/4} t^{-5/4}\Gamma(5/4), \tag{6.34}$$

where Γ is the gamma function. The integral over the other line gives an identical time dependence, so we conclude that the perturbation decays asymptotically as $t^{-5/4}$ in this case.

The contribution of the asymptotically baroclinicBC mode is simpler sincebecause there is only one singularity at $\omega = 0$ with no branch cuts. We assume that r is large enough so the asymptotic expansion of the Whittaker function can be used, $W_{\kappa,m}(r) \sim r^{\frac{\Omega}{\omega}-\frac{1}{2}}e^{-\frac{\Lambda r}{\lambda_1\lambda_2}}$ (**?**), where $\Omega = \frac{m\beta\lambda_1\sqrt{\lambda_1\lambda_2}}{2\Lambda}$. The inverse Laplace transform is then

$$\frac{1}{2\pi}\int_{-\infty+i\gamma}^{\infty+i\gamma} W_{\kappa,m}(r)d\omega \propto \frac{1}{2\pi}\int_{-\infty+i\gamma}^{\infty+i\gamma} r^{\frac{\Omega}{\omega}}e^{-i\omega t}d\omega = i\delta(t) + \frac{i\sqrt{i\Omega}}{\sqrt{t}}J_1\left(2\sqrt{|\Omega|t\ln r}\right), \tag{6.35}$$

where the last equality is from Ref. **?**. Since fFor large times, $J_1(2\sqrt{|\Omega|t\ln r}) \sim t^{-1/4}\cos(2\sqrt{|\Omega|t\ln r} - 3\pi/4)$ (**?**), so the BC mode oscillates while its amplitude decays as $t^{-3/4}$.

7. Aspects of contour-contour and contour-topography instabilities

7.1. *Contour-contour instability*

In the contour-contourCC resonance, the instability is due to the interaction of the PV waves at the liquid contours $r = R_1$ and $r = R_2$. In this case, the bottom topography at $r > R_2$ can be neglected; this amounts to setting $\beta = 0$ where it appears explicitly in (4.9)–(4.11) ~~(~~[but not setting $\beta = 0$ in the expressions for PV discontinuities Δ_1 and Δ_2 in (4.2)~~)~~]. By (4.11), ~~in this case~~ η vanishes in this case; a system of two homogeneous algebraic equations for α_1 and α_2 is established. This system can be written in matrix form,

$$\begin{bmatrix} M_{11} - \frac{\omega}{m} & M_{12} \\ M_{21} & M_{22} - \frac{\omega}{m} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{7.1}$$

where

$$M_{11} = \frac{\bar{V}_1(R_1) - \Delta_1 G_{11}(R_1, R_1)}{R_1}, \quad M_{12} = -\frac{\Delta_1 G_{12}(R_1, R_2)}{R_2}, \tag{7.2}$$

$$M_{21} = -\frac{\Delta_2 G_{21}(R_2, R_1)}{R_1}, \quad M_{22} = \frac{\bar{V}_2(R_2) - \Delta_2 G_{22}(R_2, R_2)}{R_2}. \tag{7.3}$$

~~In order t~~To have a nontrivial solution, the determinant of the 2×2 matrix in (7.1) should be zero. This yields the eigenvalue equation, which is quadratic in ω~~,~~:

$$\frac{\omega^2}{m^2} - (M_{11} + M_{22})\frac{\omega}{m} + M_{11}M_{22} - M_{12}M_{21} = 0, \tag{7.4}$$

from which we get the dispersion relation for the two-contour~~s~~ subsystem~~,~~:

$$\omega_{\mathrm{A,B}} = \frac{m}{2}\left[(M_{11} + M_{22}) \pm \sqrt{(M_{11} + M_{22})^2 - 4(M_{11}M_{22} - M_{12}M_{21})}\right]. \tag{7.5}$$

The subscript ~~A~~A or ~~B~~B corresponds to applying a + or − sign before the square root in (7.5), respectively. The two eigenvectors corresponding to the two eigenvalues in (7.5) are the two modes of PV perturbations at the liquid contours, which we ~~accordingly~~ call type ~~A~~A or type ~~B~~B; they are connected to the contour~~s'~~ deformations via (4.6) (see Refs. ??).

Figure 5 shows an example of how the CC instability can be recognized via the dispersion curves $\omega(m)$. The basic flow parameters are $R_1 = R_2 = 2.5$, $\Gamma_1 = -1$, $\lambda_1 = \lambda_2 = 1/2$; for future reference, this setup is called "configuration A" of the flow. The eigenvalues of the isolated CC system are calculated ~~via~~by using (7.5), while the eigenvalues of the full system are calculated numerically as explained in §4.2. ~~For~~To facilitate ~~easier~~ tracking of the dispersion relation, we calculate the dispersion curves for continuously varying m and mark the points corresponding to integer m, which are the physically relevant values [see (3.3)]. Figure 5a shows the angular phase velocity of the perturbations, $\mathrm{Im}(\omega)/m$ ~~vs~~versus the wave number m, for the two CC waves and the unstable perturbation. When ~~$m = 3, 4$~~$m = 3,\ 4,$ or 5, the two phase velocities of the two CC waves coincide and CC instability occurs, as can be seen from the curve for growth rate [$\mathrm{Im}(\omega)$] ~~curve~~ in ~~f~~Figure 5b.

At $m = 6$, the angular velocities of the two CC waves ~~are~~ differ~~ent~~, ~~therefore~~so no ~~there cannot be a~~ CC instability is possible. However, ~~since~~because the flow ~~is still~~remains unstable at $m = 6$ (the growth rate is nonzero), ~~the~~we conclu~~sion is~~de that one of the CC waves is in resona~~tes~~nce with the topographic perturbations at $r > R_2$ (~~since~~because the outer region at $r > R_2$ by itself is always stable, see §6).

~~In case of~~For a CC instability, the growth rate of the full system g_F can be compared ~~to~~with that found by using the isolated CC system~~,~~ g_{CC}. ~~In~~For the case shown in ~~f~~Figure 5b, at ~~the~~ low mode number~~s~~ ($m = 3$ or $m = 4$), the inequality $g_F \le g_{CC}$ holds; therefore, at these mode numbers, the topography outside causes a reduction in the growth rate. At $m = 6$ the inequality is ~~in~~reversed, $g_F > g_{CC} = 0$, the topography at $r > R_2$ then destabilizes the flow, since without it the flow would stay stable. This result is not specific to the particular parameters of the flow in this example~~, and~~ but ~~was observed~~occurred in all our calculations: at low mode numbers the CC resonance is dominant, yet the full-system growth rate is lower than expected ~~due to~~because of the CC interaction alone. Also, at larger mode numbers the CT resonance becomes the only one that contributes to instability while the CC subsystem is stable. Another example ~~for~~of this result is given below.

7.2. *Contour-topography instability*

As ~~is~~ shown in ~~f~~Figure 5, the unstable $m = 6$ mode, which is not caused by a CC resonance, has a real angular velocity very close to that of one of the CC-interaction modes. This suggests that ~~actually~~ the CC perturbation type B, having the lowest angular velocity of the two modes ~~(~~[see (7.5)], is actually the ~~one~~mode that resonates with the topographic perturbations ~~at~~for $r > R_2$. ~~In order t~~To identify the resonating perturbation type in the CC system by the integral eigenvalue approach, we rewrite the eigenvalue equations (4.9)–(4.11) ~~in a way~~so that the CC perturbation types appear decoupled; the calculation is given in Appendix F. This procedure can be viewed as partial diagonalization of the system of equations (4.9)–(4.10) by moving to the CC eigenmode~~s~~ coordinates. ~~t~~The resulting equations (G 5)–(G 7) in Appendix F are diagonal in the isolated CC system (i.e., in case ~~that~~ there is no topography outside the contours). ~~It is found~~The results indicate that, ~~in case of~~for the CCT instability, only type B is in resona~~tes~~nces with the topographic perturbations.

To understand why type B is the ~~one~~mode that resonates with the topographic perturbations, we apply pseudomomentum considerations. Recall that two modes may resonate only if their pseudomomenta are of opposite ~~in~~ sign (§5.1). ~~Since~~Because the topographic types have positive pseudomomentum [see (5.4)], only the type having negative pseudomomentum can resonate with them. ~~In a~~Appendix F ~~it is~~ prove~~d~~s that the pseudomomentum of a perturbation has the same sign as the slope of the dispersion curve (when m may be taken to vary smoothly); a similar result was ~~proven~~obtained for ~~the rotating~~ shallow, rotating water (one layer, zonal) ~~case~~ in Ref. ?. ~~From~~ ~~f~~Figure 5a~~, it is~~ shows clearly that, at $m = 6$, only type B has ~~the~~ negative pseudomomentum ~~since~~because ~~only~~ its dispersion curve is the only one that decreas~~ing~~es with m at $m = 6$.

As ~~is~~ shown in ~~f~~Figure 2b, a flow with parameters $R_1 = 5$, $R_2 = 2$, $\Gamma_1 = -1$, $\lambda_1 = \lambda_2 = 0.5$ (we ~~denote~~call this flow configuration ~~by~~ B) is unstable ~~to~~against mode $m = 2$ perturbations, where the instability is the CT instability. Figure 6 shows that ~~in this case~~ $m = 2$ is the**[AU: You may want to explain what is meant by "gravest unstable mode," or use a different terminology.]**gravest unstable mode in this case. Again, the full-system phase velocity is close to that of perturbation type ~~*B*~~B of the CC subsystem~~;~~ ~~this~~, which is ~~in accordance~~consistent with its decreasing dispersion curve, pointing the fact that it ~~bears~~has negative pseudomomentum (Appendix F).

Figure 7 shows the growth rates of different mode numbers as functions of the radius R_2 of the lower ring ~~R_2~~ for the basic flow parameters $R_1 = 5$, $\lambda_1 = \lambda_2 = 0.5$, and $\beta = -0.1$. When $\Gamma_1 = 1$ (~~f~~Figure 7a), the lines of the $m \geq 2$ modes are composed of two "bulges" that get close to each other with increasing mode number until the~~ir~~y merg~~ing~~e at $m = 8$. The instability in th~~ese~~is ~~'~~bulge~~'~~ regime is of type CC, as ~~was~~ shown for the $m = 2$ case in ~~f~~Figure 2a. ~~At~~On the left ~~side~~ of each of the lines, it becomes nearly horizontal; in this range the instability is of type C_2T (this ~~was~~is also shown for ~~the~~ $m = 2$ ~~case~~ in ~~f~~Figure 2a). Between these two regions the instability is of type CCT. ~~M~~Unlike the other modes, mode $m = 1$~~, contrary to the other modes,~~ is unstable only ~~due to~~because of the CC resonance.

Similarly, when $\Gamma = -1$ (~~f~~Figure 7b), the lines of the growth rates ~~at~~for $m \geq 2$ modes

Figure 5: Real and imaginary parts of the eigenvalues for CC and full resonance. The basic flow parameters are $R_1 = 2.5$, $R_2 = 2.5$, $\Gamma_1 = -1$c, $\beta = -0.5$ (configuration A). (a) Perturbation angular velocity $\mathrm{Re}(\omega)$ ~~vs~~versus the mode number m. The angular velocities of two CC waves are given by red and blue dotted curves, and that of the full system is given by the green curve whenever there is instability. Points with physically relevant values of m (integers) are marked. S, CC, and CT designate regions where the flow is stable, unstable ~~due to~~because of CC resonance, and unstable ~~due to~~because of CT resonance, ~~accordingly~~respectively. (b) ~~The g~~Growth rate $\mathrm{Im}(\omega)$ ~~vs~~versus the mode number for the CC resonance (g_{CC}, purple) and for the full system (g_F, green). The points are joined by straight lines for better visualization.

Figure 6: Real and imaginary parts of the eigenvalues for CC and full resonance. The basic flow parameters are $R_1 = 5$, $R_2 = 2$, $\Gamma_1 = -1$, $\beta = -0.1$ (configuration B). (a) Perturbation angular velocity $\mathrm{Re}(\omega)$ ~~vs~~versus mode number m. (b) Growth rate $\mathrm{Im}(\omega)$ ~~vs~~versus mode number for the full system. ~~N~~The notation and colors are the same as in Figure 5.

Figure 7: Growth rates $\mathrm{Im}(\omega)$ for different mode numbers as functions of R_2 at $R_1 = 5$, $\beta = -0.1$, $\Lambda = 1$, and $\lambda_1 = \lambda_2 = 0.5$ for (a) $\Gamma_1 = 1$, (b) $\Gamma_1 = -1$. ~~T~~Each curve is labeled by its mode number~~s are labeled next to each curve~~.

~~can be seen as~~are composed of three parts: one is the low-R_2 regime, where the lines are nearly horizontal, ~~and then~~in which case the instability is of type $\mathrm{C_1T}$. The CC instability part consists of the line where a steep increase in growth rate begins (going from left to right). Between these two regions the instability is of type CCT; this was also shown for the $m = 2$ case in ~~f~~Figure 2b. Again, mode $m = 1$ is unstable only ~~due to~~because of CC resonance.

7.3. *Barotropic and baroclinic contour-topography resonance*

Another useful property of the eigenvalue equations in integral form, (4.9)–(4.11), is the simple separation of ~~barotropic~~BT and ~~baroclinic~~BC couplings. By equations (3.15)–(3.18), the Green's functions G_{11}, G_{12}, G_{21}, and G_{22} are linear combinations of the two more basic~~,~~ ~~baroclinic~~BC and ~~barotropic~~BT Green functions, G_{BT} and G_{BC}. The latter serve as ~~the~~ coupling coefficients between the contours' perturbations α_1 and α_2 ~~to~~and the perturbation ~~outside,~~ η outside. Therefore, if we ~~set~~use $G_{\mathrm{BT}} \equiv 0$ ($G_{\mathrm{BC}} \equiv 0$) in the integral terms in (4.9)–(4.11), only ~~baroclinic~~BC (~~barotropic~~BT) couplings to the outside perturbation are allowed. ~~O~~Upon comparing the resulting growth rates ~~in~~for any case we can identify which of the couplings is dominant. When the BT (BC) coupling is dominant, the contours ~~are~~enter in resonance with the asymptotically BT (BC) mode. If ~~they are~~ both couplings are dominant, the contours ~~are~~enter in resona~~ting~~nce with a mixed mode.

~~It was found that i~~In most cases, the ~~barotropic~~BT CT resonance ~~is the~~ dominates~~ one~~, ~~while~~whereas the ~~baroclinic~~BC CT resonance is very weak or absent. ~~A~~Figure 8a shows an example of the growth rates of the full system, the CC subsystem, the BT coupling, and the BC coupling~~ is shown in figure 8a~~; the relative thickness λ_1 of the upper layer ~~λ_1,~~ is varied. In this case, $R_1 = 5$, $R_2 = 2$, $\Gamma_1 = -1$, $\beta = -0.1$, and the mode number is $m = 2$~~. In this case~~, so the instability is of type CCT (see ~~f~~Figure 2b)~~,~~ and the type-B perturbation of the CC subsystem enters resona~~tes~~nce with the BT perturbation type whenever an instability occurs.

An example ~~for~~of a configuration where the ~~baroclinic~~BC coupling ~~is the~~ domina~~nt~~tes ~~one~~ is not easily found because, for all r and r', the inequality $G_{\mathrm{BC}}(r,r') < G_{\mathrm{BT}}(r,r')$ holds, as ~~is~~may be verified directly from ~~equations~~ (B 6) and (B 12)~~;~~. ~~m~~Moreover, if $|r-r'| \gg r$, this inequality gets stronger: $G_{\mathrm{BC}}(r,r') \ll G_{\mathrm{BT}}(r,r')$. Therefore, the ~~baroclinic~~BC interaction terms are usually negligible compared ~~to~~with the ~~barotropic~~BT terms. ~~T~~As a result, the growth rates and the eigenfunctions are~~, accordingly,~~ determined mainly by the ~~barotropic~~BT couplings. This explains the ~~barotropic~~BT governor effect, in which ~~barotropic~~BT shear ~~leads to~~ reduc~~tion in~~es the ~~baroclinic~~BC growth rate (**?**). In the case of circularly symmetric flow, **?** noted that this effect may be attributed to ~~the~~ ~~barotropic~~BT strain rather than to shear, but ~~anyway~~ both the shear $\partial\bar{V}_2/\partial r$ and strain $r\partial(\bar{V}_2/r)/\partial r$ are nonzero in ~~our~~the present case.

The only ~~way to find an~~ example where the ~~baroclinic~~BC CT resonance ~~is~~ domina~~nt~~tes is ~~to look at configurations~~ whe~~re~~n the ~~barotropic~~BT growth rates ~~are close to~~approach zero. In this ~~way~~case, the subsystem with only ~~barotropic~~BT couplings can be viewed as almost stable, ~~while~~and the ~~baroclinic~~BC couplings can be viewed as small perturbation~~;~~ ~~these~~s that can affect the resulting eigenvalues of the full system. ~~S~~Figure 8b shows such an example, where $R_1 = R_2 = 5$, $\Gamma_1 = 1$, and $\beta = -0.1$~~, is shown in figure 8b~~. In the range $0.123 \le \lambda_1 \le 0.15$ the instability is of type $\mathrm{C_2T}$, ~~and then~~in which case the growth rate of the full system g_F is very close to that due to the BC coupling g_{BC}.

As λ_1 approaches ~~1~~unity (i.e., the lower layer becomes very thin) the BC coupling becomes dominant (~~f~~Figure 8b). This is consistent with the findings of Ref. **?** ~~(~~who investigated two-layer shallow water with a bottom topography~~)~~, and found that, ~~when~~as the depth of the lower layer decreases, the ~~baroclinic~~BC instability overtakes the ~~barotropic~~BT instability. In terms of the resona~~ting~~nce perturbations, however, this range of $\lambda_1 \approx 1$ cannot be attributed to instability with ~~the~~ ~~baroclinic~~BC perturbation~~ types, since~~ because the instability is CC ~~there~~in this range (~~as~~because $g_{CC} \neq 0$); the ~~baroclinic~~BC interaction only contributes to increasing ~~of~~ the growth rate, not to its origin. **[AU: To what does “its” refer (interaction, growth rate, perturbations)? For clarity, you may want to use the proper noun.]**

7.4. *Resonance with continuous spectrum*

Figure 9 shows e~~E~~xamples of three unstable solutions (found numerically) for η ~~are shown in figure 9~~ for three configurations of the basic flow. Figure 9a shows the basic flow profiles ~~F~~for configuration A (see §7.1 and ~~f~~Figure 5)~~ the basic flow profiles are shown in figure 9a~~, and Figures 9d and 9g show the amplitude and relative phase of the resulting unstable perturbation~~ are shown in figures 9d and 9g~~, respectively~~; since~~. Because $\omega_{\mathrm{r}} = \mathrm{Re}(\omega) < 0$ in this case while $\bar{V}_2 \ge 0$, this cannot be a critical layer instability. The instability is of type CC and the growing perturbation outside is reminiscent of the asymptotically BT mode (6.20)~~;~~. ~~due to the fact~~Given that ω is complex, the alternating PV profile ~~is subject to~~decreases exponential~~ decrease~~ly with r (for details see Ref. **?**).

~~T~~Figure 9b shows the PV and velocity profiles of configuration B (see §7.1 and ~~f~~Figure 6)~~ are shown in figure 9b~~. As shown above, ~~for this configuration~~ the instability is of type CCT for this configuration, and ~~since~~because $\omega_{\mathrm{r}} \in \mathcal{S}_1$, this is a critical layer instability. The unstable perturbation (~~f~~Figure 9e) is reminiscent of the critical layer structure (§4)~~; this~~, as is most clearly viewed by the rapid change in the relative phase of the perturbations ~~in~~over a thin ~~range~~region near $r_{\mathrm{c}} = m(\bar{V}_2/r)^{-1}(\omega_{\mathrm{r}})$ (~~f~~Figure 9h).

The third configuration, which we label configuration C, ~~corresponds to a case where the~~consists of a $\mathrm{C_2T}$ instability ~~is of type $\mathrm{C_2T}$~~ and a dominant ~~baroclinic~~BC coupling~~ is dominant~~. The flow parameters are $R_1 = R_2 = 5$, $\Gamma_1 = 1$, $\beta = -0.1$, $\lambda_1 = 0.14$, an~~s~~d $\lambda_2 = 0.86$~~; from f~~. Figure 8 shows clearly the dominance of the ~~baroclinic~~BC coupling~~ is~~

Figure 8: **[AU: Please label graph ordinates.]**Growth rates $\mathrm{Im}(\omega)$ for different resonances as functions of λ_1 for (a) $R_1=5$, $R_2=2$, $\Gamma_1=-1$, $\beta=-0.1$, ~~and~~ $m=2$~~,~~; (b) $R_1=R_2=5$, $\Gamma_1=1$, $\beta=-0.1$, $m=2$. ~~The~~Each curve is labeled by resonance type~~s are labeled next to each curve~~. ~~I~~The inset in the upper~~-~~ left ~~corner~~ o~~n~~f panel (b) ~~represents~~shows the growth rates in the range $0.12<\lambda_1<0.16$.

Figure 9: Examples of (a)–(c) profiles of the basic flow~~((a)-(c)) and~~, (d)–(f) the corresponding unstable PV perturbations amplitude~~((d)-(f))~~, and (g)–(i) the relative phases ~~((g)-(i))~~. The basic flow parameters for each ~~triple are:~~vertical column are (a), (d)~~+~~, (g) $R_1=R_2=2.5$, $\Gamma_1=-1$, $\beta=-0.5$, $\lambda_1=\lambda_2=0.5$ (configuration A), where $m=5$ is the gravest mode with frequency $\omega=-0.118+0.081i$. (b)~~+~~, (e)~~+~~, (h) $R_1=5$, $R_2=2$, $\Gamma_1=-1$, $\beta=-0.1$, $\lambda_1=\lambda_2=0.5$ (configuration B), where $m=2$ is the gravest mode with frequency $\omega=0.221+0.027i$. (c)~~+~~, (f)~~+~~, (i) $R_1=R_2=5$, $\Gamma_1=1$, $\beta=-0.1$, $\lambda_1=0.2$, $\lambda_2=0.8$ (configuration C), where $m=2$ is the gravest mode with frequency $\omega=-0.0097+0.0002i$. Notations and colors of ~~f~~Figures (a)–(c) are the same as in Figure 1. In panels (d)–(f) the PV perturbations in the upper layer are denoted by solid blue lines and in the lower layer by dotted blue lines. Arrows denote delta functions, their height corresponds to the ~~multiplicative factor in front~~prefactors of the delta functions.**[AU: Please verify: no dotted lines appear in Figs. 9d–9f, and no arrows appear in Fig. 9.]**

~~evident~~. The unstable perturbations (~~f~~Figures 9f~~,~~ and 9i) are reminiscent of the critical layer structure near r_c (at about 5.1) and, at $r>r_c$, it ~~is reminiscent of~~recalls the stable BC mode.

To understand the structure of the solutions ~~in the case of~~for CT instability we approximate the solution to the eigenvalue equations when the growth rates are small. The integral equation (4.11) is nonsingular and $\eta(r)$ is then given by (6.5) with no delta function and without ~~need for~~having to calculate the principal value~~ calculation, i.e.~~:

$$\eta(r)=\frac{\xi(r)}{\frac{\bar{V}_2}{r}-\frac{\omega}{m}}. \tag{7.6}$$

Plugging (7.6) into (4.11) yields the following equation for ξ~~,~~:

$$-\frac{G_{21}(r,R_1)}{R_1}\alpha_1-\frac{G_{22}(r,R_2)}{R_2}\alpha_2+\xi(r)=\int_{R_2}^{\infty}\frac{\beta G_{22}(r,r')}{\frac{\bar{V}_2(r)}{r}-\frac{\omega}{m}}\xi(r')dr'. \tag{7.7}$$

We ~~denote~~now use $\omega=\omega_\mathrm{r}+i\omega_\mathrm{i}$, where ω_r and ω_i are the real and imaginary parts of ω, respectively. If ω is near a bifurcation~~,~~ (i.e., ω_i is small), then we may assume that $\mathrm{Im}\xi$ is also small. By (3.3) the expression for the PV perturbation at $r>R_2$ is

$$q_2(r,\theta,t)=-\frac{\beta}{r}\frac{\mathrm{Re}[\xi(r)]\left(\frac{\bar{V}_2}{r}-\frac{\omega_\mathrm{r}}{m}\right)}{\left(\frac{\bar{V}_2}{r}-\frac{\omega_\mathrm{r}}{m}\right)^2+\frac{\omega_\mathrm{i}^2}{m^2}}e^{\omega_\mathrm{i}t}\cos(m\theta-\omega t), \tag{7.8}$$

where the term $\mathrm{Im}[\xi(r)]\omega_\mathrm{i}$ was neglected ~~being of~~because it is second order in ω_i. The solution vanishes at $r=m\bar{V}_2/\omega_\mathrm{r}$ and ~~switches~~changes the sign of PV between the two sides; note that the similarity to a critical layer structure is more prominent as the ratio~~n~~ $\omega_\mathrm{i}/\omega_\mathrm{r}$ becomes small.

For the basic flow in ~~f~~Figures 9b and 9c, th~~is~~e ratio is~~, accordingly,~~ $\omega_\mathrm{i}/\omega_\mathrm{r}\approx0.128$ and

$\omega_{\rm i}/\omega_{\rm r} \approx 0.1$, respectively. By (7.8), as $\omega_{\rm i} \to 0$, the ~~zone of switching~~region over which the PV changes sign~~s~~ in the unstable mode becomes more narrow; thus, in the limit of $\omega_{\rm i} \to 0$, the discontinuous nature of the critical layer is restored. In the approximation of linear perturbations, equation (7.8) describes a thin layer having an m-fold symmetry centered at $r = m\bar{V}_2/\omega_{\rm r}$ and ~~which~~that becomes stronger and broader with time.

In the case of CT resonance, ~~part~~some of the eigenmodes in the outer region $r > R_2$ ~~constitute~~form a continuum (§6). This suggests that the resonance in this case is with a collection of ~~the~~ perturbations of the continuous spectra, as was shown ~~by~~in Ref. **?**~~. A simple~~ and as may be expla~~nation to this fact may be given on grounds of~~ined simply based on pseudomomentum considerations: the pseudomomentum of the resonating perturbations in the system must ~~cancel~~sum to zero (see §5.1). Since the pseudomomentum of the contours is always finite ~~(~~[the first two terms in (5.4)~~)~~], the pseudomomentum of the topographic perturbation outside must also be finite. ~~But~~However, the pseudomomentum of one critical-layer perturbation [i.e., the third term in (5.4)~~)~~] is infinite ~~(~~by (6.5)~~); therefore in order~~, so ~~to have~~any ~~a~~ finite-pseudomomentum topographic perturbation~~, it~~ must be composed of a collection of critical-layer perturbations such that the third term in (5.4) is finite.

Following Ref. **?**, we use projections to determine the structure of this collection~~, we use projections~~. The unstable outer PV perturbation in the lower layer, $\eta_\omega(r)$ ~~(~~[where $\mathrm{Im}(\omega) > 0$~~)~~], is projected on the possible stable self-excitations of the outer region discussed in §4, $\eta_{\omega'}$ ~~(~~{where ω' ~~being~~is real~~,~~ and the critical layer is at $r_{\rm c} = (V_2/r)^{-1}(\omega'/m)$~~)~~]. ~~Since~~Because the stable solutions constitute an orthonormal set (see §4.1), the projection $\langle \eta_\omega, \eta_{\omega'} \rangle$ correctly calculates the weights in this collection. By using (6.5) we get

$$\begin{aligned} \langle \eta_\omega, \eta_{\omega'} \rangle &= \int_{R_2}^{\infty} \eta_\omega(r) \left[D(\omega')\delta\left(\frac{\bar{V}_2(r)}{r} - \frac{\omega'}{m}\right) + P\frac{\xi^*_{\omega'}(r)}{\frac{\bar{V}_2}{r} - \frac{\omega'}{m}} \right] dr \\ &= \frac{D(\omega')}{\left|(\bar{V}_2/r)'_{r_{\rm c}}\right|}\eta_\omega(r_{\rm c}(\omega')) + P\int_{R_2}^{\infty} \frac{\xi_\omega(r)\xi^*_{\omega'}(r)dr}{\left(\frac{\bar{V}_2}{r} - \frac{\omega}{m}\right)\left(\frac{\bar{V}_2}{r} - \frac{\omega'}{m}\right)} \\ &\approx \frac{D(\omega')}{\left|(\bar{V}_2/r)'_{r_{\rm c}(\omega')}\right|}\eta_\omega(r_{\rm c}(\omega')) - \frac{i\pi}{\left|(\bar{V}_2/r)'_{r_{\rm c}(\omega')}\right|}\eta_\omega(r_{\rm c}(\omega'))\xi^*_{\omega'}(r_{\rm c}(\omega')). \end{aligned} \tag{7.9}$$

The principal value integral was calculated ~~via~~by using Cauchy's integral theorem (see, e.g., Ref. **?**) ~~using~~and the fact that $\omega_{\rm i} > 0$ for an unstable mode, and by assuming that the main contribution to the integral is near the critical layer. We assume that ~~the changes in~~ $D(\omega')$, $\xi_{\omega'}(r_{\rm c}(\omega'))$, and $\bar{V}_2(r_{\rm c}(\omega'))$ ~~with~~depend weakly on ω' ~~are small~~ relative to ~~changes in~~ $\eta_\omega(r_{\rm c}(\omega'))$. In this case, the weight ~~then~~ goes as

$$|\langle \eta_\omega, \eta_{\omega'} \rangle|^{1/2} \sim |\eta_\omega(r_{\rm c}(\omega'))|^{1/2} = \left[\frac{|\xi_\omega(r_{\rm c}(\omega'))|}{(\omega' - \omega_{\rm r})^2 + \omega_i^2}\right]^{1/2}. \tag{7.10}$$

If $\xi_\omega(r_{\rm c}(\omega'))$ ~~changes slowly enough with~~depends weakly enough on ω', then ~~the expression in~~ (7.10) is maximized at $\omega' = \omega_{\rm r}$~~; this~~, which means that the resona~~ting~~nce eigenfunctions are those with frequencies close~~d~~ to $\omega_{\rm r}$. Figure 10 shows that the approximation (7.10) is ~~in excellent agreement~~consistent with the direct calculation of the weight. Approximating (7.10) further by assuming that the ~~nominator~~numerator is constant leads to

$$|\langle \eta_\omega, \eta_{\omega'} \rangle|^{1/2} \sim \left[\frac{1}{(\omega' - \omega_{\rm r})^2 + \omega_i^2}\right]^{1/2}, \tag{7.11}$$

as obtained by Ref. **?** in the ~~one-layer~~single-layer, zonal, rotating shallow-water model.

~~T~~However, this expression~~, however,~~ is not a good approximation for frequencies outside the immediate neighborhood of ω_{r}, as shown in ~~f~~Figure 10.

8. Nonlinear evolution of contour-contour versus contour-topography instabilities

According to the linear stability analysis scheme, ~~at first glance~~ it seems at first glance that ~~there shouldn't be any~~no substantial difference should exist between the evolution of the flow ~~in case of~~for a CC instability ~~to~~and that ~~of~~for a CT instability. After all, the source of the instability ~~brings~~causes the entire system to collectively rotate and grow; the unstable solutions of the linear stability analysis ~~unstable solutions~~ are written as if ~~the~~ phase-locking ~~is achieved~~occurs immediately. However, in practice, ~~the~~ phase~~-~~-locking ~~is an~~ evolv~~ing effect~~es over time (see, e.g., Ref. ?). If the system is subject to some random noise, the first two parts to phase-lock are the resona~~ting~~nce perturbation types. Therefore, ~~with~~over time, they ~~become~~grow into the dominant ~~growing~~ perturbations, ~~where~~and the ~~rest of the~~remaining perturbations are influenced by these ~~first~~initial ~~ones~~perturbations. The subsystem ~~that~~in resona~~tes~~nce ~~reaches~~is the first to have large-scale perturbations~~ first~~, ~~and consequently~~so nonlinear effects become pronounced first for this subsystem.

We ~~conduct~~now discuss high-Reynolds-number simulations ~~employing~~done with the coefficient-form partial differential equation package of ~~the~~ COMSOL, which is ~~software~~ based on the finite-element method (see Ref. ? for details). The vorticity-diffusion term $\nu\nabla^2 Q_i$ is added to the ~~right-hand side part~~RHS of ~~equations~~ (2.5)~~ in order~~ to maintain numerical stability. The resulting coupled system composed of ~~equations~~ (2.4) and the equation ~~of~~for PV evolution ~~(~~[i.e., ~~equation~~ (2.5) supplemented with the diffusion term~~)~~] is solved as an initial-value problem in a two-dimensional (r,θ) rectangular grid with the limits $1 < r < 30$ and $0 \le \theta \le 2\pi$. The unknown variables are the streamfunction and the PVs. We apply the periodicity conditions at $\theta = 0$ and $\theta = 2\pi$ and the no-slip conditions at both radial boundaries by setting $\partial\Psi_i/\partial r = \partial\Psi_i/\partial\theta = 0$ at $r = 30$, and $\partial\Psi_i/\partial r = 0$ and $\Psi_i = 0$ at $r = 1$.

The computational domain is $30 \times 2\pi$ in size and is divided into three subdomains. The first is a fine-grid domain that covers $1 \le r < 1.5$ with ~~the~~a mesh size of 0.05×0.03~~,~~; it is set off ~~in order~~ to ~~be able to~~ resolve the viscous boundary layer that may form next to the cylinder. The second is the main domain and covers $1.5 \le r < 20$ with ~~the~~a mesh size of 0.1×0.03. In both these domains, ν is set to ~~be~~ 0.0001. The third domain~~,~~ covers $20 \le r \le 30$~~,~~ and is set off as an absorbing layer to prevent reflections. ~~In order t~~To ~~get~~obtain a reasonable machine time for the evolution of linear instability of the flow, we add over the entire computational grid a random perturbation to the basic PV field in the form of Gaussian noise~~ on the entire computational grid~~. More details on the method used ~~can be found~~are available in Ref. ?.

8.1. *Contour-contour instability*

Figure 11 shows an example of the evolution of an unstable flow ~~in case o~~for~~f~~ a CC instability~~ is shown in~~. The flow parameters are the same as ~~in~~for ~~f~~Figures 5 and 9a~~,~~; namely, $R_1 = 2.5$, $R_2 = 2.5$, $\Gamma_1 = -1$, and $\beta = -0.5$. As ~~is~~ shown in ~~f~~Figure 5b, the gravest unstable mode is $m = 5$, ~~and~~which is indeed ~~this is~~ the mode that evolve~~d~~s most rapidly in the simulation. ~~At first~~Initially, the ~~deformation of the~~ contours ~~is seen~~deform at $t = 30$; the upper contour ~~is~~ tilt~~ed~~s relative to the lower ~~one~~contour, and they are phase-locked and propagate ~~in the~~ clockwise~~ direction~~ in accordance with the calculated frequency of the linear stability analysis, $\omega = -0.118 + 0.081i$.

Figure 10: Amplitude of ~~the~~ spectrum of ~~the~~ PV perturbation in ~~the~~ lower layer at $r > R_2$ (blue solid line), ~~the~~ approximate~~d~~ expression (7.10) (red dotted line), and ~~the~~ approximate~~d~~ expression (7.11) (green dotted line) for $R_1 = 5$, $R_2 = 2$, $\Gamma_1 = -1$, $\beta = -0.1$ (configuration B).

Figure 11: Evolution of ~~the~~ relative PV field in ~~the~~ upper layer (upper panel in each ~~cuple~~pair) and ~~the~~ lower layer (lower panel in each pair). The basic flow parameters are $R_1 = R_2 = 2.5$, $\Gamma_1 = -1$, $\beta = -0.1$, and $\lambda_1 = \lambda_2 = 0.5$ (configuration A); mode 5 is the most unstable. Red~~/~~ and blue ~~colours mark~~designate positive~~/~~ and negative PV, and gr~~e~~ay~~,~~ designates the island. Time is specified in ~~non-~~dimension~~al~~less units at the upper-left corner of the upper panel in each ~~couple~~pair.

As ~~is~~ shown at $t = 60$ and $t = 80$, the contour perturbations ~~of the contours~~ excite the perturbation types outside in the form of waves, which are similar to the stationary waves ~~found~~discussed in §6.2. Since the ~~baroclinic~~BC wave mode decreases exponentially with distance, the BT wave dominat~~ wave is the barotropic one~~es, which decreases only as $r^{-1/4}$. This ~~barotropic~~BT mode has the form of spirals, as was shown for ~~barotropic~~BT flows on the beta cone (**?**). These linear waves appear only in the lower layer, where the gradient of the basic PV exists.

During the nonlinear grow~~ing~~th of the deformation, five ~~couples~~pairs of partially overlapping negative (in the upper layer) and positive (in the lower layer) PV patches can be identified (most clearly at $t = 60$). These can be viewed as modons, i.e., QG ~~baroclinic~~BC vortical dipoles (see, e.g., Ref. **???**). For each ~~couple~~pair, the positive part ~~stays~~remains attached to the cylinder, ~~while~~whereas the negative part is released and moves more freely (as can be seen ~~from~~for time $t \geq 160$~~ and on~~). ~~At first~~Initially ($t = 90$), each ~~couple~~pair is created such that ~~it~~its propagat~~ion aim~~es toward~~s~~ the cylinder~~; this~~, which causes the positive patch to deform and ~~get close to~~approach the cylinder. T~~hen t~~he positive patch then ~~switches~~changes partner and the dipole moves outwards again ($t = 120$). Upon reaching a maximal distance from the island, the modons swing ($t = 160$) and return~~s back~~ to the cylinder. ~~Then~~Next, the modons collide, exchange ~~their~~ partners, and new modons emerge. ~~Due to~~Because of wave radiation, dissipation, and filamentation, ~~this time~~ the maximal distance from the cylinder is smaller this time. This process repeats in a quasi-periodic manner in a ~~similar~~ fashion similar to the ~~the~~ ~~barotropic~~BT evolution shown ~~by~~in Ref. **?**. At $t = 230$ the ~~5-~~fivefold symmetry is lost, two of the positive parts in the upper layer leave the cylinder, and the five modons are wandering around.

The evolution in this case of CC instability is very similar to ~~evolution~~that of unstable ~~barotropic~~BT flows on the beta cone studied by **?**~~;~~. ~~t~~The main features of the QG evolution are~~:~~ the emergence of modons (instead of dipoles in the ~~barotropic~~BT case), which ~~having a~~ tend~~ency~~ to move counterclockwise~~,~~; the appearance of spiral ~~barotropic~~BT PV waves propagating clockwise~~,~~; and a quasiperiodic outward and inward motion of ~~the~~ modons ~~with~~that exchange ~~exchanging~~ partners every cycle. The "averaged" beta in this system can be defined according to the weight of each layer as $\lambda_1 \cdot 0 + \lambda_2 \cdot \beta = \lambda_2 \beta$, ~~so it is~~which gives -0.25 for the evolution in ~~f~~Figure 11. As expected and as ~~was~~ found in simulations (data not shown), the maximal distance of the modons increases for weaker $|\beta|$, whereas, for stronger $|\beta|$, new flow patterns form without the emergence of modons ~~for stronger $|\beta|$~~.

Figure 12: Evolution of ~~the~~ relative PV field in the upper layer (upper panel in each ~~cuple~~pair) and ~~the~~ lower layer (lower panel in each pair). The basic flow parameters are $R_1 = 5$, $R_2 = 2$, $\Gamma_1 = -1$, $\beta = -0.1$, and $\lambda_1 = \lambda_2 = 0.5$ (configuration B); mode 2 is the most unstable. Colors and notation~~s~~ are the same as in ~~f~~Figure 11.

8.2. *Contour-topography instability*

~~While~~Whereas the CC instability ~~results in~~produces flow evolutions ~~which are~~ analogous to th~~at~~ose in the ~~barotropic~~BT case, the CT instability is rather different. The main reason is that ~~now~~ the dominant interaction is now between one of the contours and a perturbation at $r > R_2$. This perturbation ~~is reminiscent of~~recalls the critical layer solution ~~as shown~~ discussed in §7.4, so the contour is in resona~~tes~~nce with a thin layer ~~having alternating~~of PV with alternating sign~~s~~ located ~~at~~ some distance from it. The ~~resonating~~ parts in resonance are th~~e ones~~ose ~~having~~with the greatest ~~amplitude of~~ PV-perturbation amplitude and ~~therefore~~thus are the first to reach ~~a~~ nonlinear saturation during the phase-locking stage (cf. Ref. **?**).

~~E~~Figure 12 shows the evolution of CCT instabilit~~y~~ies ~~is shown in figure 12~~. The flow is in configuration B, as ~~in~~for ~~f~~Figures 6 and 9b, where $R_1 = 5$, $R_2 = 2$, $\Gamma_1 = -1$, and $\beta = -0.1$. The resonance is between type-B perturbations of the CC subsystem and the topographic perturbations ~~at~~for $r > R_2$. As shown in Figure 7b, the CCT instability in this configuration is close to the $\mathrm{C_1T}$-instability regime, so the mutual deformation of the contours significantly deforms C_1 ($t = 30$), whereas C_2 suffers only minor deformations. By $t = 30$, a narrow PV ring with $m = 2$ symmetry forms at $r \approx 4.2$; this ring is the collection of critical-layer perturbations with critical layers in the vicinity of $r = 4.2$ (§7.4).

Out of the initial random perturbations that were inserted into the system, only the resonant perturbations begin to phase-lock and grow. Therefore, the C_1 deformation and the new thin PV ring are the first to grow significantly in this case and reach large-scale perturbations, where nonlinearity becomes important. Nonlinear effects stop the linear growth and the thin ring rearranges in the configuration shown at $t = 90$. This cessation of growth explains why no dipolar modons emerge, contrary to the CC-instability case (§8.1), where both contours are significantly deformed.

From $t = 90$ to $t = 230$, the flow rotates counterclockwise and completes about three revolutions in a quasi-stationary manner. This structure is a ~~baroclinic~~BC version of the tripolar structure found in the ~~barotropic~~BT beta-cone model (**?**, Figure 14) and recalls the stationary two-layer QG tripole vortices found numerically on the f plane (**??**) and that were also investigated on the beta plane (**?**).

The tripole eventually breaks into two modon quartets ($t = 260$) composed of two PV patches at each layer. In the upper layer, the PVs of the circle and its adjacent patch are equal, approximately -1, as was initially the case. In the lower layer, the PVs differ because only the PV of the circular patch was there initially, its value being approximately 8.32. ~~t~~The PV of the adjacent patch, which ~~has~~ emerge~~d~~s from the interaction with the upper-layer PV, is 0.3 on average. ~~Since~~Because the positive-PV circular core ~~at~~in the lower layer is so strong and only the non~~-~~circular patch in the upper layer is tilted vertically relative to it, this quartet behaves effectively as a dipolar modon (composed of ~~the~~a circular positive vortex in the lower layer and a noncircular vortex patch in the upper layer). Therefore, after reaching a maximal distance of about 2.8 from the island ($t = 260$), the modon swings and ~~comes back~~returns to the cylinder ($t = 294$).

9. Conclusion

We ~~have~~ investigate~~d~~ herein the possible resonances leading to instability of two-layer QG circular flows around an island with the sea bottom sloping offshore ($\beta < 0$). The flow in each layer is composed of one uniform relative PV ring: the outer radius of the upper (lower) ring is R_1 (R_2) and the ~~non~~dimensionless PV inside it is $\Gamma_1 = +1$ or -1 [Γ_2, given by (4.3)~~)~~]. ~~The~~An azimuthal normal-mode analysis leads to a set of integral eigenvalue equations ~~which~~that have direct physical interpretation in terms of the possible resonances of the system.

T~~he possible t~~opographic PV perturbations ~~exist~~are possible only in the lower layer at $r > R_2$, where a nonzero PV gradient occurs. A continuous set of possible perturbations consists of those having a critical layer. Asymptotically, a~~t~~s $r \to \infty$, these solutions split into two ~~kinds~~types~~,~~: ~~barotropic~~BT or ~~baroclinic~~BC modes. When these modes rotate clockwise, they are wavelike in the radial direction, so~~ and therefore on the two-dimensional plane~~ a pattern of spiral PV patches appear on the two-dimensional plane. ~~Both modes, a~~Although both modes seem to be neutral in the normal analysis scheme, the full initial-value treatment shows that they actually decay ~~with~~over time ~~when the full initialvalue treatment is considered~~.

At low mode numbers (usually $m = 2$, 3, 4 for the cases ~~surveyed in this paper~~discussed herein), the CC resonance ~~is~~ domina~~n~~tes ~~over~~ the CT interactions, yet the full-system growth rate is ~~lower~~less than expected ~~due to~~because of the CC interaction alone. Thus, ~~the~~ coupling to the external topographic perturbations ~~causes~~ stabiliz~~ation of~~es the system. At larger mode numbers, the CT interaction becomes the only unstable ~~one~~interaction.

For a fixed radius ~~of the upper layer ring~~ R_1 of the upper layer ring, the radius R_2 of the lower layer ring ~~R_2~~ determines the type of resonance that leads to instability. When the lower ring is sufficiently thin ~~enough~~ (i.e., R_2 ~~is close to~~approaches 1), the dominant resonance is CT~~;~~. ~~i~~If $\Gamma_1 = +1$ (in which case the flow in both layers is clockwise), then ~~it~~the resonance is specifically C_2T, ~~i.e.~~which means that the lower ring contour is in resona~~tes~~nce with the topographic perturbations ~~existing~~ outside of it. If $\Gamma_1 = -1$ (in which case the flow in both layers is counterclockwise), then ~~it~~the resonance is specifically C_1T, ~~i.e.~~which means that the upper ring contour is in resona~~tes~~nce with the topographic perturbations at $r > R_2$. The transition from small R_2, where the instability is C_1T (or C_2T), to large R_2, where the instability is CC, occurs through the CCT instability. In this instability, one of the perturbation types of the CC subsystem enters resona~~tes~~nce with the topographic perturbations, but not C_1 (or C_2) itself.

The resonance ~~of~~between the contours ~~with~~and the topographic perturbations may be ~~either~~ dominated either by ~~barotropic~~BT or ~~baroclinic~~BC couplings, which are easy to identify by using the integral-equation approach. Usually, the ~~barotropic~~BT couplings ~~are~~ dominate (the ~~barotropic~~BT governor effect), but for a narrow region of the upper layer of relative thickness λ_1, the dominant instability is ~~of~~ due to baroclinity. In this case, the resonance of the contours is primarily with the asymptotic ~~baroclinic~~BC topographic mode.

The nature of the instability reflects ~~on~~ the stage of nonlinear evolution ~~stage~~ of the flow. In case of CC instability, the two contours change significantly during the phase-locking stage~~; this~~, which leads to modon~~s~~ formation and emission from the island. In the CT instability the resonance ~~is with~~involves a collection of topographic perturbations ~~types having~~with critical layers in proximity to one another. The result is a strengthening of the PV in a narrow ring at some distance ~~at~~in the lower layer. This ring interacts with the contours to form a quasistationary structure (e.g., a tripole) and only ~~at~~ later~~ times the structure~~ breaks into modons ~~which~~that may be emitted from the island.

With some minor modifications, the beta-cone concept can be ~~applied~~used to ~~the~~ treat~~ment of~~ flows in the presence of the conical beta effect ~~i~~on a planetary scale, namely, ~~of~~for the Antarctic Circumpolar Current. In this case, ~~the equation~~ (2.2) for the upper-layer PV is supplemented with an additional background planetary beta term $\beta_{\mathrm{P}} r$ (related to the gradient of the Coriolis parameter), ~~while~~and β in the lower layer is replaced by $\beta_{\mathrm{P}} + \beta_{\mathrm{T}}$ (β_{T} being related to the bottom topography); see, e.g., Ref. ?. In this case more resonances come into play, ~~since~~because more types of perturbation~~ type~~s are added in the upper layer. These issues will be considered separately elsewhere.

Acknowledgments

The author thanks Professor Z. Kizner for valuable discussions on this study. This research was supported by the US–Israel Science Foundation (BSF), Grant No. 2014206.

Appendix A. Velocity profile of basic flow

Consider the basic flow, in which the PV in each of the layers is given by (4.1). The equations can be decoupled by using the following definitions of the BT and BC modes of the basic flow (cf. ?)~~,~~:

$$\bar{Q}_{\mathrm{BT}} = \lambda_1 \bar{Q}_1 + \lambda_2 \bar{Q}_2, \quad \bar{\Psi}_{\mathrm{BT}} = \lambda_1 \bar{\Psi}_1 + \lambda_2 \bar{\Psi}_2, \quad \beta_{\mathrm{BT}} = \lambda_2 \beta, \tag{A 1}$$

$$\bar{Q}_{\mathrm{BC}} = \bar{Q}_1 - \bar{Q}_2, \quad \bar{\Psi}_{\mathrm{BC}} = \bar{\Psi}_1 - \bar{\Psi}_2, \quad \beta_{\mathrm{BC}} = -\beta. \tag{A 2}$$

From (A 1) and (A 2) we obtain

$$\bar{Q}_1 = \bar{Q}_{\mathrm{BT}} + \lambda_2 \bar{Q}_{\mathrm{BC}}, \quad \bar{\Psi}_1 = \bar{\Psi}_{\mathrm{BT}} + \lambda_2 \bar{\Psi}_{\mathrm{BC}}, \tag{A 3}$$

$$\bar{Q}_2 = \bar{Q}_{\mathrm{BT}} - \lambda_1 \bar{Q}_{\mathrm{BC}}, \quad \bar{\Psi}_2 = \bar{\Psi}_{\mathrm{BT}} - \lambda_1 \bar{\Psi}_{\mathrm{BC}}, \quad \beta_2 = \beta_{\mathrm{BT}} - \lambda_1 \beta_{\mathrm{BC}}. \tag{A 4}$$

~~U~~By using (A 3) and (A 4) along with (2.4), we arrive at the equations that relate the modal PVs and streamfunctions~~,~~:

$$\bar{Q}_{\mathrm{BT}} = \nabla^2 \bar{\Psi}_{\mathrm{BT}} + \beta_{\mathrm{BT}} r, \tag{A 5}$$

$$\bar{Q}_{\mathrm{BC}} = \nabla^2 \bar{\Psi}_{\mathrm{BT}} - \tilde{\Lambda}^2 \bar{\Psi}_{\mathrm{BC}} + \beta_{\mathrm{BC}} r, \tag{A 6}$$

where $\tilde{\Lambda} = \Lambda / \sqrt{\lambda_1 \lambda_2}$. For definiteness, we assume that $R_2 > R_1$; otherwise the following expressions should be adapted in a straightforward manner. ~~U~~By using (A 5) and (4.1), the ~~barotropic~~BT streamfunction satisfies the equation

$$\bar{\Psi}''_{\mathrm{BT}} + \frac{1}{r} \bar{\Psi}'_{\mathrm{BT}} + \beta_{\mathrm{BT}} r = \begin{cases} \lambda_1 \Gamma_1 + \lambda_2 \Gamma_2, & R \le r \le R_1 \\ \lambda_2 \Gamma_2, & R_1 < r \le R_2 \\ \beta_{\mathrm{BT}} r, & R_2 < r. \end{cases} \tag{A 7}$$

The general solution to (A 7) is

$$\bar{\Psi}_{\mathrm{BT}} = \begin{cases} -\frac{1}{9} \beta_{\mathrm{BT}} r^3 + \frac{1}{4} (\lambda_1 \Gamma_1 + \lambda_2 \Gamma_2) r^2 + C_1 \ln(r) + C_2, & R \le r \le R_1 \\ -\frac{1}{9} \beta_{\mathrm{BT}} r^3 + \frac{1}{4} \lambda_2 \Gamma_2 r^2 + C_3 \ln(r) + C_4, & R_1 < r \le R_2 \\ C_5 + C_6 \ln r, & R_2 < r. \end{cases} \tag{A 8}$$

~~It is known that t~~The expression for the energy of the flow is (see, e.g., Ref. ?)

$$E = \frac{1}{2} \iint_{r>R} [\lambda_1 (\nabla \bar{\Psi}_1)^2 + \lambda_2 (\nabla \bar{\Psi}_2)^2] r dr d\theta + \frac{1}{2} \Lambda^2 \iint_{r>R} (\Psi_1 - \Psi_2)^2 r dr d\theta, \tag{A 9}$$

~~in which~~where the first integral ~~represents~~gives the kinetic energy and the second ~~one~~integral gives the available potential energy of the flow. The potential energy is associated with the ~~baroclinic~~BC mode only, whereas the kinetic energy is contributed by both modes. Therefore, ~~in order~~ for the kinetic energy to be finite, the contribution to the first integral in (A 9) from the ~~barotropic~~BT mode should be finite and we must set $C_6 = 0$ in (A 8). By (A 7) the ~~barotropic~~BT streamfunction is continuous, ~~as well~~ as is its first derivative ~~i.e.~~:

$$\bar{\Psi}_{\mathrm{BT}}(R_1^-) = \bar{\Psi}_{\mathrm{BT}}(R_1^+), \quad \bar{\Psi}_{\mathrm{BT}}(R_2^-) = \bar{\Psi}_{\mathrm{BT}}(R_2^+), \tag{A 10}$$

$$\bar{\Psi}'_{\mathrm{BT}}(R_1^+) = \bar{\Psi}'_{\mathrm{BT}}(R_1^-), \quad \bar{\Psi}'_{\mathrm{BT}}(R_2^+) = \bar{\Psi}'_{\mathrm{BT}}(R_2^-). \tag{A 11}$$

From equations (A 10) and (A 11) the four unknowns ~~$C_1 - C_4$~~C_1–C_4 are ~~found to be~~

$$C_1 = (1/3)\lambda_2\beta R_2^3 - (1/2)\Gamma_1 R_1^2\lambda_1 - (1/2)\lambda_2\Gamma_2 R_2^2, \tag{A 12}$$

$$\begin{aligned} C_2 = {} & (1/2)\ln(R_1)R_1^2\Gamma_1\lambda_1 - (1/4)R_1^2\Gamma_1\lambda_1 - (1/3)\ln(R_2)\lambda_2\beta R_2^3 \\ & +(1/2)\ln(R_2)\lambda_2\Gamma_2 R_2^2 + (1/9)\lambda_2\beta R_2^3 - (1/4)\lambda_2\Gamma_2 R_2^2 + C_5, \end{aligned} \tag{A 13}$$

$$C_3 = (1/3)\lambda_2\beta R_2^3 - (1/2)\lambda_2\Gamma_2 R_2^2, \tag{A 14}$$

$$\begin{aligned} C_4 = {} & -(1/3)\ln(R_2)\lambda_2 * beta R_2^3 + (1/2)\ln(R_2)\lambda_2\Gamma_2 R_2^2 \\ & +(1/9)\lambda_2\beta R_2^3 - (1/4)\lambda_2\Gamma_2 R_2^2 + C_5 ~~,~~. \end{aligned} \tag{A 15}$$

Using (A 1), (A 2), and (A 8), the azimuthal ~~barotropic~~BT velocity is ~~via (A 1)-(A 2) and (A 8)~~

$$\bar{V}_{\mathrm{BT}} \equiv \frac{\partial \bar{\Psi}_{\mathrm{BT}}}{\partial r} = \begin{cases} -\frac{1}{3}\beta_{\mathrm{BT}} r^2 + \frac{1}{2}(\lambda_1\Gamma_1 + \lambda_2\Gamma_2)r + \frac{C_1}{r}, & R \le r \le R_1 \\ -\frac{1}{3}\lambda_2\beta_2 r^2 + \frac{1}{2}\lambda_2\Gamma_2 r + \frac{C_3}{r}, & R_1 < r \le R_2 \\ 0, & R_2 < r. \end{cases} \tag{A 16}$$

The velocity is assumed to vanish at $r = R$; i.e., $\bar{V}_{\mathrm{BT}}(R) = 0$ (see §4.1). ~~Using~~Applying (A 16) ~~this~~ imposes ~~a~~the relation between Γ_1 and Γ_2 ~~which~~that appears in (4.3). ~~Due to~~Based on the Stokes theorem this is equivalent to the condition of vanishing ~~the~~ total ~~barotropic~~BT excess PV (i.e., the PV that result~~ings~~ ~~by~~from omitting the background PV) in the two rings: $\int_R^{R_2} r\nabla^2\bar{\Psi}_{\mathrm{BT}}dr = \lambda_1\int_R^{R_1} r\Gamma_1 dr + \lambda_2\int_R^{R_2} r(\Gamma_2 - \beta r)dr = 0$.

~~Using~~The use of (A 6) and (4.1) shows that the ~~baroclinic~~BC streamfunction satisfies the equation

$$\bar{\Psi}''_{\mathrm{BC}} + \frac{1}{r}\bar{\Psi}'_{\mathrm{BC}} - \tilde{\Lambda}^2\bar{\Psi}_{\mathrm{BC}} + \beta_{\mathrm{BC}} r = \begin{cases} \Gamma_1 - \Gamma_2, & R \le r \le R_1 \\ \beta_1 r - \Gamma_2, & R_1 < r \le R_2 \\ \beta_{\mathrm{BC}} r, & R_2 < r. \end{cases} \tag{A 17}$$

The general solution to (A 17) is

$$\bar{\Psi}_{\mathrm{BC}} = \begin{cases} D_1 K_0(\tilde{\Lambda}r) + D_2 I_0(\tilde{\Lambda}r) - (\Gamma_1 - \Gamma_2)/\tilde{\Lambda}^2 + is_{2,0}(i\tilde{\Lambda}r)\beta_{\mathrm{BC}}/\tilde{\Lambda}^3, & R \le r \le R_1 \\ D_3 K_0(\tilde{\Lambda}r) + D_4 I_0(\tilde{\Lambda}r) + \Gamma_2/\tilde{\Lambda}^2 - is_{2,0}(i\tilde{\Lambda}r)\beta_2/\tilde{\Lambda}^3, & R_1 < r \le R_2 \\ D_5 K_0(\tilde{\Lambda}r) + D_6 I_0(\tilde{\Lambda}r), & R_2 < r, \end{cases} \tag{A 18}$$

where $s_{2,0}$ is the Lommel function s of order $\{2, 0\}$ (?). For the energy (A 9) to be finite, we must set $D_6 = 0$. The ~~barotropic~~BT streamfunction satisfies the continuity conditions at $r = R_1$ and $r = R_2$, the continuity conditions of its derivative at these radii ~~(~~[the ~~corresponding~~ equations corresponding to (A 10) and (A 11) for the ~~baroclinic~~BC mode~~)~~], and ~~its vanishing~~the condition that it must vanish at $r = R$. By solving these five equations and using the relations

$$\frac{d}{dr}K_0(\tilde\Lambda r) = -\tilde\Lambda K_1(\tilde\Lambda r), \quad \frac{d}{dr}I_0(\tilde\Lambda r) = \tilde\Lambda I_1(\tilde\Lambda r), \quad \frac{d}{dr}s_{2,0}(i\tilde\Lambda r) = -\frac{1}{2}i\tilde\Lambda \pi L_1(\tilde\Lambda r), \tag{A 19}$$

(L_1 being the modified Struve function (?)), the expressions for ~~$D_1 - D_5$~~ D_1–D_5 can be found (not given here). The azimuthal ~~baroclinic~~ BC velocity is then

$$\bar V_{\rm BC} \equiv \frac{\partial \bar\Psi_{\rm BC}}{\partial r} = \begin{cases} -D_1\tilde\Lambda K_1(\tilde\Lambda r) + D_2\tilde\Lambda I_1(\tilde\Lambda r) + \pi L_1(\tilde\Lambda r)\beta_{\rm BC}/\tilde\Lambda^2, & R \le r \le R_1 \\ -D_3\tilde\Lambda K_1(\tilde\Lambda r) + D_4\tilde\Lambda I_1(\tilde\Lambda r) - \pi L_1(\tilde\Lambda r)\beta_2/\tilde\Lambda^2, & R_1 < r \le R_2 \\ -D_5\tilde\Lambda K_1(\tilde\Lambda r), \quad R_2 < r. \end{cases} \tag{A 20}$$

The basic velocity in each layer is then found by using ~~the equations~~

$$V_1 = V_{\rm BT} + \lambda_2 V_{\rm BC}, \quad V_2 = V_{\rm BT} - \lambda_1 V_{\rm BC}, \tag{A 21}$$

~~that~~ which follow from the ~~above~~ definitions above (A 1), (A 2), (A 16), and (A 20).

Appendix B. Barotropic and baroclinic Green's functions

The ~~barotropic~~ BT Green's function $G_{\rm BT}(r, r')$ is defined by ~~the equation~~

$$\frac{d^2 G_{\rm BT}(r,r')}{dr^2} + \frac{1}{r}\frac{dG_{\rm BT}(r,r')}{dr} - \frac{m^2}{r^2}G_{\rm BT}(r,r') = \delta(r - r'), \tag{B 1}$$

and satisfies the boundary conditions

$$G_{\rm BT}(r = R, r') = 0, \quad G_{\rm BT}(r \to \infty, r') = 0. \tag{B 2}$$

The general solution to (B 1) is

$$G_{\rm BT} = \begin{cases} ar^m + br^{-m}, & R \leqslant r < r' \\ cr^m + dr^{-m}, & r' < r. \end{cases} \tag{B 3}$$

Imposing the boundary conditions (B 2) we get $b = -aR^{2m}$ and $c = 0$. By (B 1) the Green's function is continuous at $r = r'$,

$$G_{\rm BT}(r'^{+}, r') = G_{\rm BT}(r'^{-}, r'). \tag{B 4}$$

Integration of (B 1) in the neighborhoods of the singularity $r = r'$ yields

$$G_{\rm BT}(r'^{+}, r') - G_{\rm BT}(r'^{-}, r') = 1. \tag{B 5}$$

Using (B 4) and (B 5), the coefficients a, b, and d in (B 3) are found. ~~t~~The solution is

$$G_{\rm BT}(r,r') = \begin{cases} \frac{r'^{-m+1}}{2m}(R^{2m}r^{-m} - r^m), & R \leqslant r \leqslant r' \\ \frac{r'^{-m+1}(R^{2m} - r'^{2m})}{2m}r^{-m}, & r' < r. \end{cases} \tag{B 6}$$

In the same manner, the ~~baroclinic~~ BC Green's function $G_{\rm BC}$ is defined by the equation

$$\frac{d^2 G_{\rm BC}(r,r')}{dr^2} + \frac{1}{r}\frac{dG_{\rm BC}(r,r')}{dr} - \frac{m^2}{r^2}G_{\rm BC}(r,r') - \frac{\Lambda^2}{\lambda_1\lambda_2}G_{\rm BC}(r,r') = \delta(r - r'), \tag{B 7}$$

and satisfies the boundary conditions

$$G_{\rm BC}(r = R, r') = 0, \quad G_{\rm BC}(r \to \infty, r') = 0. \tag{B 8}$$

The general solution to (B 7) is (denoting $\tilde{\Lambda} = \Lambda/\sqrt{\lambda_1\lambda_2}$)

$$G^m_{\mathrm{BC}} = \begin{cases} \tilde{a}K_m(\tilde{\Lambda}r) + \tilde{b}I_m(\tilde{\Lambda}r), & R \leqslant r \leqslant r' \\ \tilde{c}K_m(\tilde{\Lambda}r) + \tilde{d}I_m(\tilde{\Lambda}r), & r' < r. \end{cases} \tag{B 9}$$

Imposing the boundary conditions (B 8), we get $\tilde{b} = -\tilde{a}K_m(\tilde{\Lambda}R)/I_m(\tilde{\Lambda}R)$ and $\tilde{d} = 0$. By (B 7) the Green's function is continuous at $r = r'$,

$$G_{\mathrm{BC}}(r'^{+}, r') = G_{\mathrm{BC}}(r'^{-}, r'). \tag{B 10}$$

Integration of (B 7) in the neighborhoods of the singularity $r = r'$ yields

$$G_{\mathrm{BC}}(r'^{+}, r') - G_{\mathrm{BC}}(r'^{-}, r') = 1. \tag{B 11}$$

Using (B 10) and (B 11) and the identity $I_m K_{m+1} + I_{m+1}K_m = 1/r$ (**?**), we get the solution

$$G_{\mathrm{BC}}(r, r') = \begin{cases} r'[I_m(\tilde{\Lambda}R)K_m(\tilde{\Lambda}r) - I_m(\tilde{\Lambda}r)K_m(\tilde{\Lambda}R)]\frac{K_m(\tilde{\Lambda}r')}{\tilde{\Lambda}K_m(\tilde{\Lambda}R)}, & R \leqslant r \leqslant r' \\ r'[I_m(\tilde{\Lambda}R)K_m(\tilde{\Lambda}r') - I_m(\tilde{\Lambda}r')K_m(\tilde{\Lambda}R)]\frac{K_m(\tilde{\Lambda}r)}{\tilde{\Lambda}K_m(\tilde{\Lambda}R)}, & r' < r. \end{cases} \tag{B 12}$$

We note that both $\frac{G_{\mathrm{BT}}(r,r')}{r'}$ and $\frac{G_{\mathrm{BC}}(r,r')}{r'}$ are symmetric with respect to switching of the variables r and r'; this fact is used in §6.

Appendix C. Pseudomomentum continuity equation

Substituting (4.6) into (3.2) gives

$$\frac{\partial s_1}{\partial t} + \frac{\bar{V}_1}{r}\frac{\partial s_1}{\partial \theta} + \frac{1}{r}\frac{\partial \psi_1}{\partial \theta} = 0, \qquad \frac{\partial s_2}{\partial t} + \frac{\bar{V}_2}{r}\frac{\partial s_2}{\partial \theta} + \frac{1}{r}\frac{\partial \psi_2}{\partial \theta} = 0. \tag{C 1}$$

Multiplying both equations ~~of~~ (C 1) by $s_i\frac{dQ_i}{dr}$ and integrating azimuthally ~~we get~~gives

$$\frac{1}{2}\frac{d\bar{Q}_1}{dr}\frac{\partial}{\partial t}\int_0^{2\pi} s_1^2 d\theta + \frac{1}{2}\frac{d\bar{Q}_1}{dr}\frac{\bar{V}_1}{r}\int_0^{2\pi}\frac{\partial s_1^2}{\partial \theta}d\theta - \frac{1}{r}\int_0^{2\pi} q_1\frac{\partial \psi_1}{\partial \theta}d\theta = 0, \tag{C 2}$$

$$\frac{1}{2}\frac{d\bar{Q}_2}{dr}\frac{\partial}{\partial t}\int_0^{2\pi} s_2^2 d\theta + \frac{1}{2}\frac{d\bar{Q}_2}{dr}\frac{\bar{V}_2}{r}\int_0^{2\pi}\frac{\partial s_2^2}{\partial \theta}d\theta - \frac{1}{r}\int_0^{2\pi} q_2\frac{\partial \psi_2}{\partial \theta}d\theta = 0. \tag{C 3}$$

The second integrals in (C 2) and (C 3) vanish identically. Multiplying (C 2) by λ_1 ~~and~~, (C 3) by λ_2, and ~~adding~~summing gives

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^{2\pi}\left(\lambda_1 r\frac{dQ_1}{dr}s_1^2 + \lambda_2 r\frac{dQ_2}{dr}s_2^2\right)d\theta = \int_0^{2\pi}\left(\lambda_1 q_1\frac{\partial \psi_1}{\partial \theta} + \lambda_2 q_2\frac{\partial \psi_2}{\partial \theta}\right)d\theta. \tag{C 4}$$

Since $q_1 = \nabla^2\psi_1 - \frac{\Lambda^2}{\lambda_1}(\psi_1 - \psi_2)$ and $q_2 = \nabla^2\psi_2 + \frac{\Lambda^2}{\lambda_2}(\psi_1 - \psi_2)$ [by (2.4)] the RHS of (C 4) ~~turns to~~takes the form (some of the integrals vanish identically),

$$\int_0^{2\pi}\left(\lambda_1\nabla^2\psi_1\frac{\partial \psi_1}{\partial \theta} + \lambda_2\nabla^2\psi_2\frac{\partial \psi_2}{\partial \theta}\right)d\theta. \tag{C 5}$$

The first term in the integral may be written as

$$\nabla^2\psi_1\frac{\partial \psi_1}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_1}{\partial \theta}\frac{\partial \psi_1}{\partial r}\right) - \frac{1}{2}\frac{\partial}{\partial \theta}\left(\frac{\partial \psi_1}{\partial r}\frac{\partial \psi_1}{\partial r}\right) + \frac{1}{2r^2}\frac{\partial}{\partial \theta}\left(\frac{\partial \psi_1}{\partial \theta}\frac{\partial \psi_1}{\partial \theta}\right), \tag{C 6}$$

and upon substitut~~io~~ing ~~of~~ (C 6) into (C 4), we get

$$-\frac{1}{2}\frac{\partial}{\partial t}\int_0^{2\pi}\left(\lambda_1 r\frac{dQ_1}{dr}s_1^2+\lambda_2 r\frac{dQ_2}{dr}s_2^2\right)d\theta+\frac{1}{r}\frac{\partial}{\partial r}\int_0^{2\pi}\left(\lambda_1\frac{\partial\psi_1}{\partial\theta}\frac{\partial\psi_1}{\partial r}+\lambda_2\frac{\partial\psi_2}{\partial\theta}\frac{\partial\psi_2}{\partial r}\right)d\theta. \tag{C 7}$$

This is the continuity equation for the pseudomomentum appearing in (5.2).

Appendix D. Differential equation for topographic perturbations at $r > R_2$

Define the operators

$$\mathrm{D}_1=\partial_\mathrm{r}^2+\frac{1}{r}\partial_\mathrm{r}-\frac{m^2}{r^2},\quad \mathrm{D}_2=\partial_\mathrm{r}^2+\frac{1}{r}\partial_\mathrm{r}-\frac{m^2}{r^2}-\frac{\Lambda^2}{\lambda_1\lambda_2}, \tag{D 1}$$

which, according to the definitions of the ~~barotropic~~BT and ~~baroclinic~~BC Green's functions (see Appendix B), satisfy

$$\mathrm{D}_1 G_\mathrm{BT}(r,r')=\delta(r-r'),\quad \mathrm{D}_2 G_\mathrm{BC}(r,r')=\delta(r-r'). \tag{D 2}$$

Define also

$$\mathrm{D}_3=\partial_\mathrm{r}^2-\frac{1}{r}\partial_\mathrm{r}-\frac{m^2}{r^2},\quad \mathrm{D}_4=\partial_\mathrm{r}^2-\frac{1}{r}\partial_\mathrm{r}-\frac{m^2}{r^2}-\frac{\Lambda^2}{\lambda_1\lambda_2}. \tag{D 3}$$

By ~~imposing~~applying the operator $\mathrm{D}_1\mathrm{D}_2$ ~~on~~to both sides of (6.9), using (3.18) and the identity

$$\int_{R_2}^{\infty}\delta^{(k)}(r)f(r)dr=(-1)^k\int_{R_2}^{\infty}\delta(r)f^{(k)}(r)dr, \tag{D 4}$$

we obtain the following ~~a~~ fourth-order nonhomogeneous differential equation ~~is achieved,~~:

$$\begin{aligned}\mathrm{D}_1\mathrm{D}_2\xi(r)=&-\frac{\mathrm{D}_1\mathrm{D}_2G_{22}(r,r_\mathrm{c})}{r_\mathrm{c}}\\&+\beta\lambda_2\mathrm{D}_4\left[\frac{1/r-1/r_\mathrm{c}}{\frac{\bar V_2(r)}{r}-\frac{\omega}{m}}\xi(r)\right]+\beta\lambda_1\mathrm{D}_3\left[\frac{1/r-1/r_\mathrm{c}}{\frac{\bar V_2(r)}{r}-\frac{\omega}{m}}\xi(r)\right],\end{aligned} \tag{D 5}$$

where the source term is proportional to $\delta(r-r_\mathrm{c})$ and its derivatives,

$$\begin{aligned}\mathrm{D}_1\mathrm{D}_2G_{22}(r,r_\mathrm{c})&=\mathrm{D}_2\mathrm{D}_1\lambda_2G_\mathrm{BT}(r,r_\mathrm{c})+\mathrm{D}_1\mathrm{D}_1\lambda_1G_\mathrm{BC}(r,r_\mathrm{c})\\&=\lambda_2\mathrm{D}_2\delta(r-r_\mathrm{c})+\lambda_1\mathrm{D}_1\delta(r-r_\mathrm{c})\\&=\mathrm{D}_1\delta(r-r_\mathrm{c})-\frac{\Lambda^2}{\lambda_1}\delta(r-r_\mathrm{c})\\&=\delta''(r-r_\mathrm{c})+\frac{\delta'(r-r_\mathrm{c})}{r}-\left(\frac{m^2}{r^2}+\frac{\Lambda^2}{\lambda_1}\right)\delta(r-r_\mathrm{c}).\end{aligned} \tag{D 6}$$

Appendix E. Poles of response function

~~In t~~This appendix ~~the~~derives the types of poles of the response function $\chi(r;r_9,\omega)$ defined by (6.30) ~~are determined~~. ~~By~~Based on (6.30), for any $\omega\neq m\bar V_2(r_0)/r_0$, χ can be written as

$$\chi(r;r_0,\omega)=\frac{1}{\frac{\bar V_2(r_0)}{r_0}-\frac{\omega}{m}}\delta(r-r_0)+X(r;r_0,\omega), \tag{E 1}$$

where $X(r,r_0;\omega)$ satisfies ~~the following equation,~~

$$\left(\frac{\bar{V}_2}{r}-\frac{\omega}{m}\right)X(r;r_0,\omega)-\beta\int_{R_2}^{\infty}\frac{G_{22}(r,r')}{r'}X(r';r_0,\omega)dr'=\beta\frac{G_{22}(r,r_0)}{\frac{\bar{V}_2(r_0)}{r_0}-\frac{\omega r_0}{m}}. \tag{E 2}$$

By (E 1), a pole exists at $\omega=m\bar{V}_2(r_0)/r_0$ ~~there is a pole~~. Equation (E 2) is singular at $r=r_c=(m\bar{V}_2/r)^{-1}(\omega)$, ~~and~~ so we use the same ansatz as in (6.1) ~~is used~~,

$$X(r;r_0,\omega)=D(r_0,\omega)\delta\left(\frac{\bar{V}_2}{r}-\frac{\omega}{m}\right)-\frac{\beta}{\frac{\bar{V}_2}{r}-\frac{\omega}{m}}\xi(r;r_0,\omega), \tag{E 3}$$

where now the last term is not defined ~~not~~ via the principal value~~,~~ and ξ is assumed to be a regular function of r. Substitut~~ion of~~ing (E 3) into (E 2) ~~results in the following equation,~~gives

$$-\xi(r;r_0,\omega)-\frac{D(r_0,\omega)G_{22}(r,r_c)}{|(V(r)/r)'_{r_c}|r_c}+\int_{R_2}^{\infty}\frac{\beta G_{22}(r,r')}{\left(\frac{\bar{V}_2(r')}{r'}-\frac{\omega}{m}\right)r'}\xi(r';r_0,\omega)dr'=\frac{G_{22}(r,r_0)}{\frac{\bar{V}_2(r_0)}{r_0}-\frac{\omega r_0}{m}}. \tag{E 4}$$

By substituting $r=R$ into (E 4), ~~the function~~ $D(r_0,\omega)$ can be expressed in terms of ξ:

$$D(r_0,\omega)=\frac{|(V(r)/r)'_{r_c}|r_c}{G_{22}(R,r_c)}\left[\xi(R;r_0,\omega)-\int_{R_2}^{\infty}\frac{\beta G_{22}(R,r')}{\left(\frac{\bar{V}_2(r')}{r'}-\frac{\omega}{m}\right)r'}\xi(r';r_0,\omega)dr'-\frac{G_{22}(R,r_0)}{\frac{\bar{V}_2(r_0)}{r_0}-\frac{\omega r_0}{m}}\right]. \tag{E 5}$$

B~~y~~ased on (E 5), $D(r_0,\omega)$ has poles along the entire segment $\mathcal{S}_1$ ~~as well as~~ in addition to the poles of $\xi(r;r_0,\omega)$. The poles of $\xi(r;r_0,\omega)$ also appear ~~also~~ in the second term ~~i~~on the RHS of (E 3).

Upon solving (E 2) for $\omega\notin\mathcal{S}_1$, a~~A~~nother class of singularities appear ~~upon solving (E 2) for $\omega\notin\mathcal{S}_1$~~. ~~Imposing~~Applying the operator $\mathrm{D}_1\mathrm{D}_2$ [where D_1 and D_2 are defined by (D 1)] ~~on~~to both sides of (E 2)~~,~~ and using (D 2) and (D 6)~~, results in the following equation,~~gives

$$\mathrm{D}_1\mathrm{D}_2\left[\left(\frac{\bar{V}_2}{r}-\frac{\omega}{m}\right)X(r,r_0;\omega)\right]-\frac{\beta\mathrm{D}_2X(r,r_0;\omega)}{r}$$
$$=\beta\frac{\delta''(r-r_0)+\delta(r-r_0)/r_0-(m^2/r^2+\Lambda^2/\lambda_2)\delta(r-r_0)~~)~~}{\frac{\bar{V}_2(r_0)}{r_0}-\frac{\omega r_0}{m}}. \tag{E 6}$$

The solution to (E 6) ~~can~~may be found ~~in the following way:~~by first~~, we~~ solv~~e~~ing the homogeneous part, which gives ~~and find~~ four linearly independent solutions. ~~Then~~Next, in any of the regions $R_2<r<r_0$ ~~and~~or $r_0<r$, the solution is written as a linear combination of the four solutions with a total~~ly~~ of ~~8~~eight constant coefficients (~~4~~four for each region). In the asymptotic limit $r\gg r_0$, ~~equation~~ (E 2) is the same as (6.1), ~~and therefore~~so only two solutions exist (i.e., the asymptotically BT and BC solutions, see §4.2). This leaves us with ~~6~~six constant coefficients. Applying (E 2) and its derivative at $r=R_2$ produces ~~There are~~ two boundary conditions ~~that can be found~~ at $r=R_2$ ~~by applying (E 2) and its derivative at $r=R_2$,~~ and four equations that match the solutions and their derivatives (up to the third-order derivative) on both sides of $r=r_0$. ~~t~~The matching conditions are determined from the delta~~-~~ terms ~~i~~on the RHS of (E 6). ~~So, there are~~This makes six (nonhomogeneous) equations for the six unknown coefficients that we designate ~~as $A_1, A_2, ...A_6$~~$A_1,A_2,\ldots,A_6$. The equations can be recast to a standard matrix notation $\mathrm{M}(r_0,\omega)\mathbf{a}=\mathbf{b}$, where $\mathbf{a}=(A_1,...,A_6)$ and $\mathbf{b}\neq\mathbf{0}$. By Cramer's rule,

the solutions are $A_i = \frac{\det(M_i(r_0,\omega))}{\det(M(r_0,\omega))}$, where $M_i(r_0,\omega)$ ~~signifies~~is the matrix formed by replacing ~~the~~column ~~i-th column~~ of M by the column vector $\mathbf{b}$. Therefore, $X(r;r_0,\omega)$ is non~~-~~analytic when the joint denominator of the coefficients, $\det(M(r_0,\omega))$, is zero. Since the determinant is a continuous function of ω, its zeros constitute a discrete set of points.

Appendix F. Sign of pseudomomentum via slope of dispersion curves

The proof here closely follows that of Ref. **?**, ~~that~~which ~~was done for~~applies to ~~one-layer~~single-layer shallow-water systems. Since the Rayleigh equation (3.4) is well defined for any $m > 0$ (not necessarily an integer) here we treat m as a continuous variable, although in practice it must be an integer [see equation (3.3)]. Multiply the Rayleigh equation (3.4) by the complex conjugate of another solution $\mathcal{Q}_1$ corresponding to a different mode number $\tilde{m}$,

$$\left(\frac{\bar{V}_1(r)}{r} - \frac{\omega}{m}\right)\mathcal{Q}_1\tilde{\mathcal{Q}}_1^* - \frac{\Phi_1}{r}\frac{d\bar{Q}_1}{dr}\tilde{\mathcal{Q}}_1^* = 0. \tag{F 1}$$

Multiply the Rayleigh equation (3.4) for the other solution by the complex conjugate of the first solution, and take the complex conjugate of the result:

$$\left(\frac{\bar{V}_1(r)}{r} - \frac{\tilde{\omega}^*}{\tilde{m}}\right)\tilde{\mathcal{Q}}_1^*\mathcal{Q}_1 - \frac{\tilde{\Phi}_1}{r}\frac{d\bar{Q}_1}{dr}\mathcal{Q}_1 = 0. \tag{F 2}$$

If the perturbation is stable, then $\tilde{\omega}^* = \tilde{\omega}$. ~~Subtracting~~The difference between the two equations ~~gives~~is

$$0 = \left(\frac{\tilde{\omega}}{\tilde{m}} - \frac{\omega}{m}\right)\mathcal{Q}_1\tilde{\mathcal{Q}}_1^* - \frac{1}{r}\frac{d\bar{Q}_1}{dr}(\Phi_1\tilde{\mathcal{Q}}_1^* - \tilde{\Phi}_1^*\mathcal{Q}_1)~~;~~, \tag{F 3}$$

$$\left(\frac{\tilde{\omega}}{\tilde{m}} - \frac{\omega}{m}\right)\mathcal{Q}_1\tilde{\mathcal{Q}}_1^* \equiv \frac{1}{r}\frac{d\bar{Q}_1}{dr}\left(\Phi_1\frac{d^2\tilde{\Phi}_1^*}{dr^2} + \frac{\Phi_1}{r}\frac{d\tilde{\Phi}_1^*}{dr} - \frac{\tilde{m}^2}{r^2}\Phi_1\tilde{\Phi}_1^* - \frac{\Lambda^2}{\lambda_1}\Phi_1(\tilde{\Phi}_1^* - \tilde{\Phi}_2^*)\right) \tag{F 4}$$
$$-\frac{1}{r}\frac{d\bar{Q}_1}{dr}\left(\Phi_1\frac{d^2\tilde{\Phi}_1^*}{dr^2} + \frac{\tilde{\Phi}_1^*}{r}\frac{d\Phi_1}{dr} - \frac{m^2}{r^2}\tilde{\Phi}_1^*\Phi_1 - \frac{\Lambda^2}{\lambda_1}\tilde{\Phi}_1^*(\Phi_1 - \Phi_2)\right).$$

Similar equations can be written for the second layer,

$$\left(\frac{\tilde{\omega}}{\tilde{m}} - \frac{\omega}{m}\right)\mathcal{Q}_2\tilde{\mathcal{Q}}_2^* \equiv \frac{1}{r}\frac{d\bar{Q}_2}{dr}\left(\Phi_2\frac{d^2\tilde{\Phi}_2^*}{dr^2} + \frac{\Phi_2}{r}\frac{d\tilde{\Phi}_2^*}{dr} - \frac{\tilde{m}^2}{r^2}\Phi_2\tilde{\Phi}_2^* + \frac{\Lambda^2}{\lambda_2}\Phi_2(\tilde{\Phi}_2^* - \tilde{\Phi}_1^*)\right)$$
$$-\frac{1}{r}\frac{d\bar{Q}_1}{dr}\left(\Phi_2\frac{d^2\tilde{\Phi}_2^*}{dr^2} + \frac{\tilde{\Phi}_2^*}{r}\frac{d\Phi_2}{dr} - \frac{m^2}{r^2}\tilde{\Phi}_2^*\Phi_2 + \frac{\Lambda^2}{\lambda_2}\tilde{\Phi}_2^*(\Phi_2 - \Phi_1)\right). \tag{F 5}$$

Multiplying (F 4) by $r\lambda_1$ and (F 5) by $r\lambda_2$, ~~adding~~summing, and then integrating with respect to r gives, after taking the limit $m \to \tilde{m}$,

$$\frac{d(\omega/m)}{dm}\int_R^\infty r\left(\lambda_1\frac{d\bar{Q}_1}{dr}|d_1|^2 + \lambda_2\frac{d\bar{Q}_2}{dr}|d_2|^2\right)dr = -\frac{2m}{r^3}\int_R^\infty \lambda_1|\Phi_1|^2 + \lambda_2|\Phi_2|^2 dr. \tag{F 6}$$

Using the definition of the pseudomomentum (5.1) ~~we get~~gives

$$\frac{d(\omega/m)}{dm}M = \frac{2m}{r^3}\int_R^\infty \lambda_1|\Phi_1|^2 + \lambda_2|\Phi_2|^2 dr. \tag{F 7}$$

The ~~right hand side~~RHS is always positive, so M has the same sign as $d(\omega/m)/dm$.

Appendix G. Rewriting eigenvalue equation in terms of contour-contour modes

The two CC perturbations types are denoted ~~by~~ A and B; each type corresponds to different perturbations of ~~the~~ contours α_1 and α_2 and different frequency~~y~~ies. We write the perturbations in vector form for ease of notation, so the eigenvectors of the CC system are

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}_A = \begin{bmatrix}\alpha_{1A}\\ \alpha_{2A}\end{bmatrix}, \quad \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}_B = \begin{bmatrix}\alpha_{1B}\\ \alpha_{2B}\end{bmatrix}, \tag{G 1}$$

with eigenvalues ω_a and ω_b, respectively. A general perturbation of the contours of the full system can be written as

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = a\begin{bmatrix}\alpha_{1A}\\ \alpha_{2A}\end{bmatrix} + b\begin{bmatrix}\alpha_{1B}\\ \alpha_{2B}\end{bmatrix}. \tag{G 2}$$

Plugging (G 2) into (4.9) and (4.10) and using the fact that the vectors in (G 1) are the CC eigenvectors, we get

$$\frac{\omega_a}{m}\alpha_{1A}a + \frac{\omega_b}{m}\alpha_{1B}b - \beta\Delta_1\int_{R_2}^{\infty}\frac{G_{12}(R_1,r')}{r'}\eta(r')dr' = \frac{\omega}{m}(a\alpha_{1A} + b\alpha_{1B}), \tag{G 3}$$

$$\frac{\omega_a}{m}\alpha_{2A}a + \frac{\omega_b}{m}\alpha_{2B}b - \beta\Delta_2\int_{R_2}^{\infty}\frac{G_{22}(R_2,r')}{r'}\eta(r')dr' = \frac{\omega}{m}(a\alpha_{2A} + b\alpha_{2B}). \tag{G 4}$$

Multiplying (G 3) by α_{2B} ~~and~~, (G 4) by α_{1B}, and subtracting ~~we get~~gives

$$\frac{\omega}{m}(\alpha_{1A}\alpha_{2B} - \alpha_{2A}\alpha_{1B})a = \frac{\omega_a}{m}(\alpha_{1A}\alpha_{2B} - \alpha_{2A}\alpha_{1B})a - \beta\Delta_1\alpha_{2B}\int_{R_2}^{\infty}\frac{G_{12}(R_1,r')}{r'}\eta(r')dr' + \beta\Delta_2\alpha_{1B}\int_{R_2}^{\infty}\frac{G_{22}(R_2,r')}{r'}\eta(r')dr'. \tag{G 5}$$

Multiplying (G 3) by α_{2A} and (G 4) by α_{1A} and subtracting ~~we get~~gives

$$\frac{\omega}{m}(\alpha_{2A}\alpha_{1B} - \alpha_{1A}\alpha_{2B})b = \frac{\omega_b}{m}(\alpha_{2A}\alpha_{1B} - \alpha_{1A}\alpha_{2B})b - \beta\Delta_1\alpha_{2A}\int_{R_2}^{\infty}\frac{G_{12}(R_1,r')}{r'}\eta(r')dr' + \beta\Delta_2\alpha_{1A}\int_{R_2}^{\infty}\frac{G_{22}(R_2,r')}{r'}\eta(r')dr'. \tag{G 6}$$

The third equation results from substituting (G 2) into (4.11),

$$\frac{\omega}{m}\eta(r) = -\frac{G_{21}(r,R_1)}{R_1}(a\alpha_{1A} + b\alpha_{1B}) - \frac{G_{22}(r,R_2)}{R_2}(a\alpha_{2A} + b\alpha_{2B}) + \frac{\bar{V}_2(r)}{r}\eta(r) - \beta\int_{R_2}^{\infty}\frac{G_{22}(r,r')}{r'}\eta(r')dr'. \tag{G 7}$$