**Students’ Conceptions of Congruent and Similar Triangles’ Definitions**

**Abstract**

There has been relatively little research on knowledge about teaching and learning of theorems and definitions of congruent and similar triangles. This study, involving 120 students, addresses high school students’ conceptions of mathematical definitions of congruent and similar triangles. The findings indicate that many of the participants differentiated between definitions and theorems and did not always accept the congruent and similar triangles theorems as formal definitions of congruency and similarity. Based on the participants’ explanations of their responses and from the interviews performed, it appears that two issues prevented some participants from accepting or preferring these theorems as definitions. The first was a concern for *uniformity*: specifically, that there is only one known and accepted definition of each concept. The second was a focus on the *essence* of the concepts: specifically, that the essence of the concepts of similarity and congruency lies primarily in the lengths of the sides of a triangles. The students who accepted these theorems as formal definitions explained their reaction as arising from the equivalence and from the theorems including necessary and sufficient attributes. The study revealed a correlation between the participants’ responses about accepting theorems as formal definitions and their reasons as evidenced in their explanations.

*Keywords*: congruent triangles theorems, formal definition, similar triangles theorems.

**1. Introduction**

Many studies have investigated how people understand the process of defining concepts and the need for definitions (e.g., Choi and Kim 2013; de Villiers 2004; Zandieh and Rasmussen 2010). Other studies have investigated how students understand the definitions of some geometric figures, such as triangles and quadrilaterals (e.g., Fujita and Jones 2007; Kaur 2015; Usiskin et al. 2008). There are also studies that have investigated students’ perceptions regarding congruent and similar triangles (e.g., Gonzalez and Herbst 2009). However, the research literature does not include any work that has clearly focused on students’ conceptions of congruent and similar triangles’ definitions. The current study attempts to fill this gap in the professional literature by examining students’ conceptions regarding the definitions of congruent and similar triangles. The results of this research contribute to our knowledge about these concepts and, in particular, to our understanding of how these definitions are characterized among students.

**1.1. Mathematical definitions.**

Many researchers have argued that definitions play a central role in mathematical theorems and proofs (Author et al. 2014; Moore 1994; Pimm 1993; Smith 2010; Weber 2002). Studies have maintained that definitions play a central role in understanding the construction of the meaning and the essence of mathematical concepts (e.g., Okazaki 2013; Wilson 1990). Okazaki (2013) reported on five ways in which fifth graders enhanced their familiarity with mathematical definitions: by understanding the meaning of the identification of geometric figures; by constructing examples from non-examples and making comparisons to justify those constructions; by recognizing equivalent combinations; by using counterexamples to evaluate undetermined cases; and by conceiving of figures as relations beyond the given actualities. Borasi (1992) claimed that definitions serve as a tool for creating uniformity in the meaning of mathematical concepts and for facilitating communication among people.

According to van Hiele and van Hiele’s (1958) theory about the development of geometric thinking, at the informal deduction level (third level), the learner understands the importance of precise definitions and of how one particular attribute derives from another. Tall and Vinner (1981) regarded definitions as the words used to specify particular concepts. Later, in Vinner's (1991) paper on the role of definitions, five assumptions were made: concepts are acquired by their definitions; students use definitions to solve problems and to prove theorems; definitions must be minimal; definitions must be elegant; and definitions are arbitrary. Zaslavsky and Shir (2005) distinguished between the roles and features of mathematical definitions, classifying the features of mathematical definitions as imperative features and optional features. The imperative characteristics they contended must exist in mathematical definitions were: the absence of any inherent contradiction between the concept attributes; the absence of ambiguity; the absence of any changes under one or another representation of the concept; and the formulation of definitions in a hierarchical (based on previous concepts) and noncircular manner. They noted that the most notable example of a controversial optional feature is the requirement that a mathematical definition be minimal. A definition is considered to be minimal if it is economical, with no superfluous or unnecessary conditions or information. In addition to the imperative criteria, Van Dormolen and Zaslavsky (2003) demanded further criteria for logical necessity: a criterion of existence; the existence of an instance of such concept; a criterion of equivalence, so that when giving more than one formulation for the same concept definition, it must be proven that all of those formulations are equivalent; and a criterion of axiomatization, which implies that a definition fits into and is part of a deductive system. They also presented two optional criteria for definition: a criterion of degeneration and a criterion of elegance, the latter holding that in a case of two equivalent definitions, the one that looks better, needs fewer words or symbols, or uses more general basic concepts should be preferred.

De Villiers (2004) distinguished between two different kinds of defining: *descriptive defining* and *constructive defining*. In descriptive defining, the image of the concept is developed before a definition is formulated for it, based on an appropriate subset of the total properties from which all of its properties can be deduced. That subset serves as a definition, and the remaining properties logically derived from it are theorems. In constructive defining, a known definition of a concept is changed through exclusion, generalization, specialization, replacement or addition in order to construct a new definition of the concept. Thus, the definition of the new concept precedes the further exploration of the concept and the development of the image of the concept.

One of the characteristics of the definitions that mathematicians and mathematical educators use is that a certain definition of a concept is equivalent to other definitions of the same concept (Harel et al. 2006; Usiskin et al. 2008). These definitions are arbitrary because they are man-made (Vinner 1991). A particular concept is defined by selecting one statement of a set of logically equivalent statements. Consequently, each of the statements in that set could be used as a legitimate definition for the particular concept.

Many studies have shown that students can have difficulties understanding the structure of definitions and their meanings (de Villiers et al. 2009; Foster 2014; Fujita and Jones 2007; Hershkowitz 1987; Marchis 2012; Pickreign 2007).

Linchevsky et al. (1992), de Villiers (1998), de Villiers et al. (2009) and Foster (2014) all reported on the tendency of students and pre-service teachers to make long lists of all the attributes of a concept. These long descriptive definitions are indeed correct, but many mathematics educators prefer mathematical definitions to be minimal and elegant, as indicated above (e.g., Leikin and Winicky-Landman 2001; Van Dormolen and Zaslavsky 2003; Vinner 1991). In contrast, there are those who, in certain cases, prefer non-minimal definitions (de Villiers 1998; Pimm 1993; Van Dormolen and Zaslavsky 2003; Zaslavsky and Shir 2005).

Leikin and Winicky-Landman (2001) investigated mathematics teachers (not in the context of geometry) and found that many high school mathematics teachers do not notice that a particular concept can be defined by a number of equivalent definitions. Vinner (1991) referred to the defining process within mathematical deductive theory as follows: “Typically, one starts with well-known notions and well-known theorems and proceeds by defining new notion and by proving new theorems” (p. 65(. Vinner (1991) added that teachers must take into account the concept acquisition and the logical reasoning that are part of this process. Harel et al. (2006) related to the difficulty of assessing the accuracy and efficiency of formal proofs and the difficulty of mathematical definitions, concluding that: “As it is commonly difficult for students to appreciate the precision and economy of thought afforded by formal proof, it is likely that they experience similar difficulty with mathematical definitions” (Harel et al. 2006, p. 153).

Van Dormolen and Zaslavsky (2003) argued that when a person offers more than one definition for a single concept, that person needs to choose one of those formulations as the definition and consider the other formulations as theorems that must be proven to be equivalent definitions of the same concept. Türnüklü et al. (2013) found that personal definitions of mathematical concepts are often based on the names given to those concepts, which can lead to many misjudgements. In addition, Author et al. (2014) reported that the term *parallelogram* affects students’ proving processes. It seems that the influence of the name on the participating students’ conceptions of definitions hindered those students from considering some of the equivalent and alternative definitions of a parallelogram as accurate definitions.

**1.2. Definitions and theorems.**

Fishbein (1994) referred to definitions and theorems as components of mathematics as a formal science. In his words: “This (the formal aspect) refers to axioms, definitions, theorems and proofs. The fact is that all these represent the core of mathematics as a formal science” (p. 231).

Freudenthal (1968) mentioned that turning definitions into theorems and theorems into definitions are some of the most fruitful activities for mathematician and the students expected to enjoy these benefits. Also, van Dormolen & Zaslavsky (2003) claimed that theorems could be considered as definitions; when there are two equivalent definitions, one can be chosen as a definition and the other formulated as a theorem.

Van Dormolen and Zaslavsky (2003) enumerated the features needed to make a mathematical definition a good definition. Among these requirements is that the definition correspond to the deductive system to which it belongs and that it be a fundamental part of that system. These deductive systems include axioms, theorems, and proofs.

Vinner (1991) mentioned that in the classroom, mathematics teachers might develop a sequence of definitions, theorems and proofs as a framework for mathematics courses. The same conceptual framework can be used with congruent and similar triangles. After proving the congruent and similar triangles theorems and highlighting the necessary and sufficient conditions of these theorems, teachers can use these theorems to solve problems. They can use them to identify, classify, and prove congruent and similar triangles. That is, these theorems fulfill the role of definitions with regard to these concepts (Moore 1994; Vinner 1991; Weber 2002).

Studies reported that students interpret the content of theorems incorrectly, with Hazzan and Leron (1996) found that students were “naive” and used theorems as vague “slogans.” In essence, such students used theorems as a way of answering test questions while avoiding the need for understanding or making excessive mental effort. In addition, Selden and Selden (2008) reported that undergraduate students often interpret the content of theorems incorrectly and have difficulty unpacking the logical structure of informally stated theorems.

**1.3. Congruent and similar triangles.**

The definition of congruent triangles is “Two triangles, △ABC and △A’B’C’ are congruent if and only if their corresponding angles are the same size and the lengths of their corresponding sides are equal.” And the definition of similar triangles is: “Two triangles, △ABC and △A’B’C’ are similar if and only if their corresponding angles are the same size and the lengths of their corresponding sides are proportional.” These definitions are non-minimal definitions. Therefore, we can mention fewer attributes and deduce the remaining attributes. There are theorems in which we mention minimal attributes in order to reach congruence triangles or similar triangles. These theorems focus on the sets of necessary and sufficient attributes that ensure congruency or similarity of triangles. For example, “Two triangles, △ABC and △A’B’C’ are congruent if and only if two angles and the inscribed side are equal.”

The concept of congruent triangles is an important part of the basic knowledge needed to teach plane geometry (Luo and Lin 2007). The congruent triangle has a significant position because it links to similarity, and because the three conditions for triangle congruency also serve as the basis for proving other propositions (Jones et al. 2013). Wu (2005) claimed that the cases of congruency and similarity highlight the need for definitions; without a mathematical definition of congruence and without a precise definition of similarity, learners cannot properly understand other topics in geometry, such as length and area.

Gonzalez and Herbst (2009) proposed the following four concepts about congruency: the perceptual conception of congruency; the measure-preserving conception of congruency; the correspondence conception of congruency; and the transformation conception of congruency. We can use these same concepts and adjust them for similarity.

Jones and Fujita (2013) reported that many eight grade students in Japan had not fully developed their correspondence conceptions of congruency. They added that about 40% of those students were not sure how to use congruent triangles to deduce conclusions.

Many studies have investigated students’ perceptions of congruent triangles theorems (Hadas et al. 2000; Hoyles 1998; Jones et al. 2013). These studies examined students’ understanding that the conditions in the congruent and similar triangles theorems are actually necessary and sufficient conditions for producing and constructing congruent or similar triangles. In the congruent and similar triangles theorems, necessary and sufficient attributes are used to deduce other attributes. These theorems can serve as formal definitions for congruent and similar triangles, as they include all of the imperative and optional features of mathematical definitions, such as parsimony and elegance (Van Dormolen and Zaslavsky 2003; Zaslavsky and Shir 2005). For example, it is possible to define two similar triangles as two triangles that are similar if and only if two angles of one triangle are congruent to the corresponding two angles of the other triangle. Based on those attributes, we can deduce the four remaining attributes, which also exist in similar triangles. An understanding of these theorems strengthens the understanding of mathematics as deductive theory and as well as the logical necessities of mathematical context (Okazaki 2013; Van Dormolen and Zaslavsky 2003; Vinner 1991). However, from a pedagogical perspective, adherence to this minimal definition only may impair students’ understanding of the concept of similar triangles (Zaslavsky and Shir 2005).

We have not found any studies in the research literature that have clearly focused on the definitions of congruent and similar triangles among students. If the theorems related to these concepts can function as definitions, why is this so? Identifying the reasons for this phenomenon could shed light on how students perceive the conceptions of the definitions of congruent and similar triangles’ definitions.

**1.4. Research rationale and goals.**

When learning that the attributes included in these theorems are sufficient to construct two similar or congruent triangles, some students go on to learn the proofs of these theorems, indicating that they are aware that these theorems contain necessary and sufficient attributes. Furthermore, the existence of more than one theorem might underscore the equivalence between these theorems. These processes can clarify the logical structure of the mathematical definition, the elegance of these definitions (when fewer words and symbols are used) and the minimalism of these definitions (when minimal attributes are used in the proofs).

However, no studies could be found in the literature that have clearly focused on the definitions of congruent and similar triangles, the relationship between congruent triangles theorems and similar triangles theorems, or how these concepts are defined by students. One of the abilities expected at the fourth level of van Hiele and van Hiele’s (1958) hierarchy is the ability to understand definitions, axioms, theorems, and proofs as connected units in a deductive structure. Therefore, the explicit goal of this study is to investigate the students’ conceptions of congruent and similar triangles’ definitions. The reasons for students’ acceptance or non-acceptance of theorems of congruent and similar triangles as definitions of those concepts could provide insights into the characteristics of mathematical definitions as perceived by students.

Tasks related to congruent and similar triangles were chosen for a number of reasons. First, these concepts are very familiar to the participants; they learn them at the junior high school level, close to the start of their lessons about proofs and deduction. Second, the logical structure that exists between the attributes of these concepts makes them easy subjects with which to conduct this type of study. Finally, there are very famous and useful theorems related to these concepts. Since students are expected to use those theorems in identification, construction and proving tasks, this study sought to ascertain whether these tasks affected the participants’ conceptions about congruent and similar triangles’ definitions.

**2. Method**

The study reported here addresses high school students’ conceptions regarding the definitions of congruent triangles and similar triangles. It focuses on how these definitions and their logical structures (i.e., that each definition contains necessary and sufficient attributes) are understood. This study aims to answer the questions of: how the participants defined the similar triangles and congruence triangles concepts; and what the characteristics of the definitions of congruent and similar triangles were, according to the participants.

**2.1. Participants.**

The research sample consisted of students from a regional Arab high school in central Israel. The sample included 120 out of 340 tenth grade students in the school, who studied geometry with four different teachers in four parallel groups at the four-points level. (In Israel, there are three levels in the matriculation exams in mathematics: three, four and five points, the latter considered the most demanding level). Two of the teachers had a bachelor’s degree in mathematics and two of them had a master’s degree in mathematics education. All had more than fifteen years’ experience in teaching mathematics. The participating students studied units covering all the congruent triangles theorems and the first similar triangles theorem (angle, angle). In ninth and tenth grades, students studied the other two similar triangles theorems (angle, side, angle; side, side, side).

**2.2. Instruments and procedure.**

The research instruments included a two-staged questionnaire (Appendix 1) developed especially for this study, as well as semi-structured interviews. In the first stage, the students were asked to define the congruent and similar triangles concepts. The second stage of the questionnaire examined the participants’ perceptions of the mathematical definitions of congruent and similar triangles. The first stage of the questionnaire was constructed to scan the knowledge students had acquired about the definition of congruent triangles and similar triangles concepts. The construction of the second stage of the questionnaire and the interviews following it were based on the analysis of students’ responses from the first stage. This analysis provided the opportunity to examine aspects of the issues which had not previously been evident. For example, we had not known that the majority of the students who gave minimal definitions of the congruent and similar triangles concepts based their definitions on sides only. Consequently, included in the second stage of the questionnaire was one minimal definition based only on sides and another minimal definition based only on angles. The second stage of the questionnaire also included two tasks: one concerning congruent triangles, and the other concerning similar triangles (see Fig 1.)

|  |
| --- |
| Figure 1. The Second Stage of the Questionnaire1. Two students debated how similar triangles should be defined. Sami said, “Two triangles, △*ABC* and △*A’B’C* are similar if and only if their corresponding angles are the same size and the lengths of their [corresponding sides](https://en.wikipedia.org/wiki/Corresponding_sides) are [proportional](https://en.wikipedia.org/wiki/Proportionality_%28mathematics%29).” Rami argued that Sami’s definition included a superfluous condition and suggested the following definition: “Two triangles, △*ABC* and △*A’B’C’* are similar if and only if they have two congruent angles.”

 Which definition/s is/are correct? Explain your answer!1. Two students debated how to define congruent triangles. Sami said, “*Two triangles, △ABC and △A’B’C’ are congruent if and only if their corresponding angles are the same size and the lengths of their* [*corresponding sides*](https://en.wikipedia.org/wiki/Corresponding_sides) *are equal.”* Rami said that there was a superfluous condition in Sami's definition and suggested the following definition: “*Two triangles, △ABC and △A’B’C’ are congruent if and only if all three of their side are equal.”* Which definition/s is/are correct? Explain your answer!
 |

In each task, two definitions were given: Sami’s definition and Rami’s definition. Both definitions were correct, but Sami’s definitions were non-minimal while Rami’s were minimal. Sami defined similar triangles as follows: “Two triangles, △ABC and △A’B’C’ are similar if and only if their corresponding angles are the same size and the lengths of their corresponding sides are proportional.” Sami defined congruent triangles as follows: “Two triangles, △ABC and △A’B’C’ are congruent if and only if their corresponding angles are the same size and the lengths of their corresponding sides are equal.”

Rami’s minimal definitions included necessary and sufficient attributes to lead to all of the critical attributes to which Sami referred. Furthermore, Rami’s definitions were based on congruent triangles or similar triangles theorems. Rami defined similar triangles as follows: “Two triangles, △ABC and △A’B’'C’ are similar if and only if they have two congruent angles.” Rami defined congruent triangles as follows: “Two triangles, △ABC and △A’B’C’ are congruent if and only if all three of their sides are equal.”

The two tasks in the second stage of the questionnaire, one based in theorem and including only sides and the second including only angles, were chosen in order to investigate whether this difference would be a factor in the participants’ responses. Specifically, the goal was to investigate whether the participants would accept a definition that included only sides and reject a definition that related only to angles. In these tasks, the participants were asked to reflect on the proposed answers, which gave them the opportunity to use critical thinking. In addition, the participants were asked to explain their responses, their explanations revealing some of their views and knowledge regarding definitions and theorems. These explanations and the interviews that followed the completion of the questionnaire helped advance the research goal of ascertaining the participants’ conception of the definitions of congruent and similar triangles.

The tasks in this study are representative tasks only; other tasks could have been designed based on other congruent triangles and similar triangles theorems. For example, if we had wanted participants to engage in another task related to the concept of similar triangles, we could have asked whether the statement, “three sides of one triangle are proportional to three sides of other triangle,” is a definition of similar triangles. We could also have asked whether the statement, “two congruent angles and the included sides of one triangle are equal to the corresponding parts of the other” is a definition of congruent triangles.

The first stage of the questionnaire was distributed during one mathematics lesson. The second stage of the questionnaire was distributed two weeks after collecting the first stage of the questionnaire and analyzing it. All but three students volunteered to complete the questionnaires. These three students had a low achievement level in mathematics. They asked their teachers for permission not to participate in the study, and their teachers granted their requests. The remaining participants completed answering the first stage of the questionnaire within 10 minutes and the second stage of the questionnaire within 15 minutes.

After administering the questionnaires and analyzing the responses, I interviewed eleven participants who provided answers and explanations that were representative of the difficulties reported by the majority of participants. For example, some of the interviewed participants did not accept the two theorems of congruent and similar triangles as formal definitions of those concepts. Others didn’t accept the similar triangles theorem which included only angles as part of its formal definition, but accepted the congruent triangles theorem that included three equal sides as part of its formal definition. Each interview lasted about 17 minutes. The structured part of the interview included the same questions that had been asked in the questionnaire, and the unstructured part included questions formulated according to the interviewees’ responses in the structured part. The goal of the interview was to determine whether the participants were indeed certain of their answers and to clarify points that were not addressed by the questionnaire or which required deeper examination. For example, the questionnaire sought to examine whether the participants would accept a minimal definition of similar triangles that included only angles as a formal definition, but did not ask whether the participants accepted other minimal definitions of the same concept which included only sides. The interviews provided the opportunity to delve into these follow-up issues, thereby adding important nuance to the questionnaire findings. This method was chosen for the purpose of addressing trends and tendencies that could arise from the questionnaire results in a setting in which the participants could be directly approached in a more focused manner (two representative interviews are presented as Appendix 2).

 **2.3. Data analyses.**

The students’ responses were analyzed using both qualitative and quantitative methods. For analyzing the explanations about the student’s responses, a qualitative coding method was utilized (Salanda 2015) that is close to grounded theory (Glaser & Strauss, 1967). Deductive codes derived from a theoretical perspective were employed (Charmaz et al. 2007), as well as inductive codes for the themes not present in existing research about geometric education. Using the deductive codes, the answer of each participant was characterized according to its satisfaction of the aspects of definition. The construction of categories was concluded when the students’ responses presented no new categories. All of the codes from the questionnaires were entered into the SPSS program and frequencies were calculated. Next, a Pearson chi-squared test was performed to see whether there was any statistical significance for the relation between the students’ explanations and their answers to accepting definitions.

3. Results

This section, describing participants’ answers in detail, is based on an analysis of the three tasks presented in Stage 1 and Stage 2 of the questionnaire.

**3.1. Stage 1, Task 1: definition of congruent and similar triangles concepts.**

In Task 1 of the first stage of the questionnaire, participants were asked to define the concepts of congruent triangles and similar triangles concepts. Our analysis revealed five categories of responses to Task 1, as described below:

1. ***Examples of non-economical definition***:
* Two triangles are similar if and only if their corresponding angles are the same size and the lengths of their corresponding sides are in the same proportion.
* Two congruent triangles are similar if their corresponding angles are the same size and the lengths of their corresponding sides are equal.
1. ***Examples of economical definition including only sides***:
* Two triangles are similar if their sides are in the same proportion.
* Two triangles are congruent if all three of their side are equal.
1. ***Examples of economical definition including angles:***
* Two triangles are congruent if two of their sides are equal and the inscribed angle between them is equal.
* Two triangles are similar if two of their side are proportional and the inscribed angle between them is equal.
1. ***Example of non-sufficient definition:***
* Two triangles are congruent if their angles are equal.
1. ***Examples of intuitive definitions:***
* Two congruent triangles are similar if each of the triangles covers the other.
* Two triangles are similar if they have the same shape but are different in size.

Table 1

*Responses to Task 1*

|  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- |
|  | Non- economical definition | Economical definition including only sides | Economical definition including angles | Non-sufficient definition  | Intuitive definition | No responses | Total  |
|  |  |  |  |  |  |  |  |
| Congruent triangle concept  | 6957.5% | 4033.33% | 54.16% | - | 32.5% | 32.5% | 120100% |
| Similar triangle concept | 6755.83% | 3529.16% | 32.5% | 86.66% | 43.33% | 32.5% | 120100% |

In Table 1, we can see a clear tendency for giving non-economical definitions for congruent and similar triangles concepts, with 57.5 % giving a non-economical definition for congruent triangles concept and approximately 56% giving a non-economical definition for the similar triangles concept. In addition, after examining the students who gave economical definitions and comparing them with the students who gave economical definitions including angles, it was found that the vast majority of the former gave economical definitions which included only sides (about 33% who gave definitions for the congruent triangles concept and about 29% who gave definitions for the similar triangles concept). Moreover, it can be seen that very few students gave intuitive definitions.

**3.2. Stage 2, Task 2:** **definition based on the similar triangles theorem (angle, angle).**

In Task 2, participants were asked to choose between Sami’s non-minimal definition that two triangles are similar if and only if their corresponding angles are the same size and the lengths of their corresponding sides are proportional, and Rami’s definition that two triangles are similar if and only if they have two congruent angles. Our analysis revealed six categories of explanations about responses to Task 1 and Task 2, as described below:

1. ***Examples of difference between definition and theorem***:
* Sami’s argument is a definition and Rami’s is a theorem and there is a difference between definition and theorem.
* Rami used a theorem and not a definition.
1. ***Examples of uniform definition:***
* There is one accepted definition.
* Sami’s definition is the accepted one for the concept of similarity of triangles, with the necessary attributes mentioned in detail.
* This the known definition for all the students and teachers.
1. ***Examples of mathematical essence of the concept:***
* Sami gave a long definition that includes all of the conditions of congruency, but Rami’s definition is also accepted as a formal definition; it emphasizes the meaning of the concept.
* Rami’s definition is also accepted as a formal definition; it emphasizes the meaning of the concept.
* Sami describes the meaning of similarity and this is a good.
1. ***Examples of non-sufficient definition:***
* Rami includes non-sufficient attributes. He didn’t mention all the attributes.
* Rami needs to mention more attributes about similarity.
1. ***Examples of equivalent definition:***
* From equal sides we can deduce equal angles, but it is more accurate to use Sami’s definition.
* From one we can deduce the other. These are equivalent.
1. ***Necessary and sufficient attributes:***
* He uses a congruence theorem. The other describes the congruent triangle and this is a good, but very long definition.
* Without a doubt, Rami is right; it is sufficient that two angles from one triangle be equal to two angles in another triangle to say they are similar triangles.

Table 2

*Responses to Task 2*

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Difference between definition and theorem | Uniform definition | Mathematical essence of the concept  | Non-sufficient definition | Equivalent definition  | Necessary and sufficient attributes  |  | Total |
| Only Sami’s definition is correct. | 1613.33% | 2420% | 1815% | 86.67% | - | 21.67% |  | 6856.67% |
| Only Rami’s definition is correct. | - | 10.83% | - | - | 32.5% | 86.67% |  | 1210% |
| Both definitions are correct. | 10.83% | 32.5% | 21.67% | 10.83% | 1714.17% | 1613.33% |  | 4033.33% |
| Total | 1714.17% | 2823.33% | 2016.67% | 97.5% | 2016.67% | 2621.67% |  | 120100% |

The Pearson chi-squared test revealed a correlation between the participants’ responses about the acceptance of the definitions and the explanations they gave for their responses (chi-square [10, N=120, p=.000<0.01]). As evident in Table 2, there was a tendency to accept only Sami’s non-economical definition for the similar triangles concept. Approximately 57% of all the participants claimed that only Sami’s non-economical definition was right, and about 35% among them said that there was a uniform definition. Of those students who claimed that only Sami’s non-economical definition was right, about 27% said that it emphasized the mathematical essence of the concept. These participants argued that the definition must reveal the mathematical essence of the concept. And 24% among the students who accepted only Sami’s non-economical definition said that there was a difference between definition and theorem.

Only about 33% among all the participants claimed correctly that both definitions were correct. About 83% among them said that the definitions were equivalent or that Rami’s definition included necessary and sufficient attributes for defining the similar triangles concept. These participants’ explanations indicate that they behaved as expected to reach van Hiele and van Hiele’s (1958) third level. For example, Tamir explained, “Sami’s definition derives from Rami’s definition*.*” Tamir understood the equivalence of the definitions and understood that the theorem of similar triangles (angle, angle, theorem) provides a minimal definition for similar triangles.

Yossif was interviewed in light of his very interesting responses. For Yossif, the equality of angles did not fully reflect the meaning of the concept of similar triangles. In the interviews, we had the opportunity to investigate whether replacing the definition based on the similar-triangles theorem (angle, angle) with the other theorem, based on the other similar triangles theorem (side, side, side) would cause to Yossif to change his response and accept the theorem based only on sides as definition for the similar triangles concept.

***Interview 1: Yossif***

Interviewer: *Can we use the criterion “two angles of one triangle have the same measure as two angles of another triangle” to identify two similar triangles?*

Yossif: Yes, we can use it and we used it in order to do tasks in geometry.

I: *In the questionnaire, you claimed that Rami’s definition […] is wrong*.

Y: Yes, Rami’s is not right definition.

I: *Although it describes similar triangles*?

Y: Yes, because it does not give us the essence and the meaning of the concept.

I: *Could the attribute “three sides are proportional in two triangles” be a classification criterion for similar triangles?*

Y: Yes, this is the theorem. And we sort similar triangles by it.

I: *One student defined similar triangles as follows:“Two triangles are similar when all of their corresponding sides have lengths of the same ratio.” Can you accept it as a correct definition?*

Y: Yes, I can accept it as a correct definition, because in this definition, the essence of the concept is clear.

I: *Does the [aforementioned] statement equivalent to the statement: “two angles of one triangle are equal to two angles of the other triangle”?*

Y: Yes, because from one theorem we can conclude the other theorem.

I: *Why did you accept one theorem as a definition and not the other?*

Y: Because of the essence of the concept. One gave us the essence and the other did not.

Yossif did not understand that all theorems of similar triangles provide a minimal definition for similar triangles. It was important for him that the definition include the attributes that embody the essence of the concept (i.e., the sides are proportional). Yossif accepted that: “three sides are proportional in two triangles*”* and “two angles of one triangle have the same measure as two angles of the other triangle” as criteria for classifying similar triangles, but he accepted only the first criterion as a definition of similar triangles. He claimed that only criteria that express the essence of the concept can constitute a formal definition. Thus, Yossif failed to reach van Hiele and van Hiele’s (1958) fourth level, at which the learner understands the function of mathematical definitions in terms of identifying and classifying examples and non-examples of a given concept. Yossif also failed to understand the concept of necessary and sufficient attributes and the equivalence of formal mathematical definitions.

To conclude, knowledge of the similar triangles theorem did not guarantee that a participant would accept it as a formal definition for similar triangles.

**3.3. Stage 2, Task 3: based on the congruent triangles theorem (side, side, side).**

In Task 3, participants were asked to choose between Sami’s definition that two triangles are congruent if and only if their corresponding angles are the same size and the lengths of their corresponding sides are equal, and Rami’s definition that two triangles are congruent if and only if they have three equal sides. The participants’ responses to Task 3 are presented in Table 3.

Table 3

*Responses to Task* 3

|  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Difference between definition and theorem | Uniform definition | Mathematical essence of the concept  | Non-sufficient definition | Equivalent definition  | Necessary and sufficient attributes  | No explanation | Total |
| Only Sami’s definition is correct.  | 1512.5% | 1210% | - | 86.67% | - | - | 10.83% | 3630% |
| Only Rami’s definition is correct.  | - | 108.33% | 75.83% | - | 21.67% | - | 10.83 | 2016.67% |
| Both definitions are correct. | - | 65% | 2520.83% | - | 119.17% | 1512.5% | 32.5% | 6050% |
| No response | 32.5% | 10.83% | - | - | - | - | - | 43.3% |
| Total | 1815% | 2924.17% | 3226.67% | 86.67% | 1310.83% | 1512.5% | 54.17% | 120100% |

The Pearson chi-squared test revealed a correlation between the participants’ responses about the acceptance of the definitions and the explanations they gave for their responses (chi-square [18, N=120, p=.000<0.01]). In Task 3, 30% (36) of the participants claimed, incorrectly, that only Sami’s non-economical definition was right. In their explanations, about 42% of them referred to the claim that there is a difference between definitions and theorems. One example is Soli’s explanation that “Sami mentions a definition and Rami mentions a theorem, and there is a difference between them.” About 33% of the students based their explanations on the ubiquity of the definition, including Sewar, who claimed that “Sami’s definition is what accepted in all the textbooks and the teachers as a definition.” About 22% of the students who claimed, incorrectly, that only Sami’s non-economical definitions was right, claimed that Rami’s definition included non-sufficient attributes. These students did not understand the meaning of the congruent triangles theorem.

Half of the participants claimed correctly that both definitions were right, and accepted the economical definition based on the congruent triangles theorem (side, side, side) as a valid definition for the congruent triangles concept. About 42% of them claimed that Rami’s definition emphasized the essence of the concept. Approximately 43% of the participants who claimed correctly that both definitions were right based their explanations on the argument that Rami’s definition included necessary and sufficient attributes.

The surprising result was that about 17% of the participants claimed incorrectly that only Rami’s economical definition based on the congruent triangles theorem (side, side, side) was right. Half of these students based their explanation on the uniformity of definition, with 35% of them explaining that their conclusions were based on the mathematical essence of the concept.

Samir was interviewed in light of his very interesting responses. In the congruent triangles and the similar triangles tasks, he answered incorrectly that only one definition was correct, because of the difference between definition and theorem. The interview provided an opportunity to pursue what Samir meant in his explanation.

***Interview 2: Samir***

Interviewer: *Did you accept the statement “two triangles, △ABC and △A’B’C’ are congruent if all the three side are equal” as a definition for congruent triangles?*

Samir:No, I can’t accept that.

I: *Although it is a theorem for congruency?*

S:Yes. Because there is a difference between definition and theorem*.*

I: *What is the difference?*

S: Sami’s is the definition and there is only one known and accepted definition; the other is theorem which one has to prove.

I: *What are the roles of the congruent triangles theorems?*

S:To distinguish congruent triangles from non-congruent triangles …and help us to prove that two triangles are congruent.

I: *So it is a base upon which to decide whether two triangles are congruent or not congruent?*

S:Yes.

I: *And it couldn’t be a definition?*

S:No, it couldn’t be a definition*.*

I: *I want to tell you that for one concept there could be more than one definition.* *The definition must contain necessary and sufficient attributes and some of the roles of definitions are to sort examples and non-examples of the concept and to serve as a base for proofs.*

S: *…..*

I: *Could you change your answer about the congruent triangles’ theorems?*

S: What is there to change?

I: *Regarding whether they could be definitions?*

S: I think yes; they can be definitions*.*

I: *And what about the similar triangles theorems?*

S:Although it difficult for me to accept the first theorem (angle, angle), but these theorems could be definitions for the similar triangles concept.

I: *Why it is difficult for you to accept the first theorem as a definition?*

S: Because it didn’t give the meaning of the concept about proportion*.*

Samir didn’t accept the theorems of congruency as formal definitions for congruency, preferring the definitions that were known and accepted. While the uniformity of definitions was an important issue for Samir, he also thought that there was a difference between definitions and theorems. Once Samir understood and recognized the roles and features of mathematical definitions, which are needed to reach van Hiele and van Hiele’s (1958) fourth level, he changed his answer and accepted the congruent triangles theorems as a definition for the concept. But he did not accept the similar triangles theorem, which includes two equal angles, as a definition for similar triangles concept because it did not provide the meaning of the proportional sides. Like Yossif, he demanded a definition that offered the meaning and the essence of the concept. Furthermore, Samir preferred the definitions that were known and accepted. The uniformity of definitions was an important issue for Samir.

**4. Discussion and Conclusions**

The present research sought to examine the conceptions students hold regarding the definitions of similar triangles and congruent triangles. Investigating the acceptance or non-acceptance of theorems of congruent and similar triangles as definitions of those concepts could provide insights about the characteristics of mathematical definitions as perceived by students.

Many of the participants isolated the defining process within mathematical deductive theory (Vinner 1991). Many did not recognize that theorems might be definitions and that congruent triangles and similar triangles theorems are formulations of the definitions of those concepts and that after they are proven and accepted as true, they become definitions that are equivalent to the non-parsimonious definitions in which all of the attributes are mentioned (Van Dormolen and Zaslavsky 2003). These students did not enjoy the benefits of transforming theorems into definitions (Freudenthal 1968). This behavior is an example of the tendency to interpret the content of theorems incorrectly and the inability to unpack the logical structure of the theorem (Hazan and Leron 1996; Selden and Selden 2008). This is reflected in the finding that participants’ knowledge of congruent triangles and similar triangles theorems did not guarantee that they would accept those theorems as formal definitions of these concepts, an ability associated with the fourth level of van Hiele and van Hiele’s (1958) hierarchy. Furthermore, accepting one theorem as a formal definition of the concept did not guarantee accepting the other theorem as a formal definition. This conforms with the findings of other studies regarding the equivalence of definitions (Harel et al. 2006; Usiskin et al. 2008). In this work, only 33% of the participants accepted the similar triangles theorem that “two triangles, △ABC and △A’B’C’ are similar if and only if they have two congruent angles” as a formal definition of similarity (see Table 2). The situation regarding the congruent triangles theorem was better, with 50% of the participants accepting the statement “two triangles, △ABC and △A’B’C’ are congruent and only if all three side are equal” as a formal definition for the congruency (see Table 3). The fact that some of the students did not accept these theorems as formal definitions indicates difficulties in understanding the characteristics, roles and features of mathematical definitions. All of the similar triangles and congruent triangles theorems specify these concepts (Tall and Vinner 1981), and although these theorems have the imperative features of mathematical definitions — there is no inherent contradiction between the concept attributes; there is no ambiguity; there are no changes under one or another representation of the concept; and the definitions are hierarchical and noncircular (Zaslavsky and Shir 2005) — these participants did not accept these theorems as formal definitions. These findings conform with those of previous studies, which revealed the tendency of students to make long lists of all of the attributes of a particular concept (de Villiers et al. 2009; Foster 2014; Linchevsky et al. 1992).

In accordance with the national curriculum, in the classroom, the students use the congruent and similar triangles theorems to solve classification, identification, and proving tasks. These theorems met the criteria of concept definitions, but many of the students did not accept these theorems as definitions. The participants were “naïve” and used these theorems without making the mental effort to consider whether the theorems could serve as definitions (Hazzan and Leron 1996).

In a comprehensive examination of the findings from Task 2 and Task 3, the Pearson chi-squared test revealed a correlation between the participants’ responses about the acceptance of the definitions and the explanations they gave for their responses (sig. = 0.000, *p* < 0.01). The students who didn’t accept the similar triangles and congruent triangles theorems as formal definitions gave explanations arguing that there was a difference between definition and theorems, because of the essence of the concept or because of the uniformity of definition. The students who accepted these theorems as formal definitions gave explanations about equivalent definitions or arguing that the theorems included necessary sufficient attributes to define the concept. This is what is expected at the formal deductive level of van Hiele and van Hiele’s (1958) hierarchy.

Zaslavsky and Shir (2005) and Van Dormolen and Zaslavsky (2003) distinguished between two kinds of features of definitions: imperative features and optional features. The current study expanded upon those models by adding another type of feature to these models: namely, a non-critical feature. This additional optional feature is the *essence* of the concept. The results indicate that the participants accepted (see Tables 2 and 3 and interview with Yossif) or preferred (as in the interview with Samir) a formal definition that emphasized the essence and the meaning of the name of the concept and, therefore, they accepted a definition that included a description of the essence of the concept (De Villiers 2004; Okazaki 2013; Wilson 1990). It could be that the equality or proportionality of the lengths of the triangles’ sides is seen as more essential to the concepts of congruency and similarity than angles are. This could explain why more participants accepted the minimal congruent triangles’ definition based on the congruent triangles theorem, which contains only sides, rather than the minimal definition of similar triangles based on the similar triangles theorem, which contains only angles. This finding confirms other research about the effects of the name of a concept on mathematical judgments (Author et al. 2014; Türnüklü et al. 2013).

The participants in a different study gave greater weight to size than they did to correspondence (Gonzalez and Herbst 2009). There is evidence that the vast majority of the students who gave or accepted economical definition based their definitions on sides only (see Table 1, Table 2 and Table 3). For example, in the interviews with Yossif, he did not accept the statement that “two triangles, △ABC and △A’B’C’ are similar if and only if they have two congruent angles” as a formal definition. But, when this statement was replaced with another similar triangles theorem, “two triangles are similar when all of their corresponding sides have lengths in the same ratio,” he accepted that theorem as a correct definition. For Yossif, the similar triangles theorems could not be used concurrently as definitions. This confirms other studies which have shown the misunderstanding of two of the characteristics of mathematical definitions: namely, that definitions are arbitrary (Vinner 1991) and that a certain definition of a concept may be equivalent to other definitions of the same concept (Harel et al. 2006; Usiskin et al. 2008).

An additional non-critical optional feature is the feature of uniformity. From the results, it can be seen that the participants accepted formal definitions based on the uniqueness of the concept definitions. They wanted to believe that for every concept, there was only one accepted definition within the mathematics education community, while all other statements were attributes. The participants understood that the subsets of conditions mentioned in the congruent and similar triangles theorems provide necessary and sufficient attributes to deduce the remaining attributes (Hadas et al. 2000; Hoyles 1998; Jones et al. 2013), but still accepted the one uniform, accepted, non-parsimonious definition. This result is congruent with those of other studies that have reported about the inability to identify, accept or find equivalent definitions (Author et al. 2014; Harel et al. 2006; Leikin and Winicky-Landman 2001).

To conclude, the students’ difficulties in understanding the characteristics and roles of mathematical definitions of geometric concepts affected their understandings of mathematical and geometric definitions. We can see evidence of this in the interview with Samir. When he understood that the definition must contain necessary and sufficient attributes and that some of the roles of definitions are to sort examples and non-examples of the concept and to serve as a base for proofs, he changed his response and accepted the theorem as a formal definition. One can argue about what a good definition is, but it can be concluded that when attributes are necessary and sufficient for classifying a concept, they can constitute a formal definition. For many participants, the essence of the mathematical concept (Mariotti and Fischbein 1997) is more important than the essence of the mathematical definition (Leikin and Winicky-Landman 2001). From a pedagogical perspective, one should not adhere to minimal definitions in the cases of congruent triangles and similar triangles because the non-minimal definitions emphasize the essence of these concepts (Zaslavsky and Shir 2005). Rather, students must understand that the minimal definitions are correct and valid definitions. This approach emphasizes the fact that mathematics is a logical science. This study’s results emphasize the need to avoid focusing only on descriptive definitions and avoid neglecting constructive definitions (de Villiers 2004). This also highlights the importance of addressing other situations reported by Okazaki (2013), in order to enhance learners’ familiarity with definitions: namely, conceiving figures as relations beyond the given actualities and recognizing equivalent combinations.

**4.1. Limitations, future directions and practical implications.**

Future studies should involve larger and more diverse research populations. Future studies should also include teachers or pre-service teachers, as well as populations from different sectors of society and different parts of the world. This would enable us to determine whether cultural differences might affect the findings. It would also be interesting to use a different methodology, such as classroom observations, to gather more qualitative information about the population under study. It would be intriguing to see what emerges within the classroom discourse during such lessons, in order to learn about the thinking processes of both teachers and students, and, most importantly, the interaction between those processes.

This study sought to investigate whether the participants accepted the congruent triangles and similar triangles theorems as formal definitions of those concepts. However, the questionnaire included only one theorem for congruency and one theorem for similarity. In the interviews with Yossif, when the similarity was replaced, he changed his response. In the future, I would like to examine the behavior exhibited by Yossif in a larger population. To that end, in future studies, it would be helpful to use a questionnaire that includes more than one congruent triangles theorem and more than one similar triangles theorem.

The results of this work may help researchers design educational studies that target particular characteristics of students’ perspectives of definitions. In conclusion, my recommendations emerging from this study’s findings are two-pronged. First, students should participate in the process of defining. Second, when in training, geometry teachers should be exposed to the specific difficulties uncovered by this study. This will raise their awareness of the processes that lead to these difficulties and sensitize them, thus helping them cope with these issues in the teaching process. Creating such a mind set and motivation will help mathematics teachers diagnose and analyze students’ difficulties and thereby perform better as teachers. Ultimately, these changes should also improve student achievement.

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Appendix 1

Questionnaires

|  |
| --- |
| First stage1. Please define the concept congruence-triangles.
2. Please define the concept similar-triangles.

Second stage1. Two students debated how similar triangles should be defined. Sami said, “Two triangles, △ABC and △A’B’C’ are similar if and only if their corresponding angles are the same size and the lengths of their [corresponding sides](https://en.wikipedia.org/wiki/Corresponding_sides) are [proportional](https://en.wikipedia.org/wiki/Proportionality_%28mathematics%29).” Rami argued that Sami's definition included a superfluous condition and suggested the following definition: “Two triangles, △ABC and △A’B’C’ are similar if and only if they have two congruent angles.”

Which definition/s is/are correct? Explain your answer!1. Two students debated how to define congruent triangles. Sami said, “Two triangles, △ABC and △A’B’C’ are congruent if and only if their corresponding angles are the same size and the lengths of their [corresponding sides](https://en.wikipedia.org/wiki/Corresponding_sides) are equal.” Rami said that there was a superfluous condition in Sami’s definition and suggested the following definition: “Two triangles, △ABC and △A’B’C’ are congruent if and only if all three of their side are equal.” Which definition/s is/are correct? Explain your answer!
 |

Appendix 2

Interviews

Interview 1: Yossif

Interviewer: *Hi, I want to ask you some question about the questionnaire you responded.*

Yossif: *O.K*

I:*How can you define similar triangles?*

Y: *Two triangles have the same angles and the sides are proportional.*

I: *This can used in order to prove two similar triangles?*

Y: *Yes.*

Interviewer: *Can we use the criterion "two angles of one triangle have the same measure as two angles of another triangle" to identify two similar triangle?*

Y: Yes, we can use it and we used it in order to do tasks in geometry.

I: *In the questionnaire, you claimed that Rami’s definition […] is wrong.*

Y: *Yes, Rami’s is not right definition.*

I: *Although it describes similar triangles?*

Y: *Yes, because it does not give us the essence and the meaning of the concept.*

I: *Could the attribute "three sides are proportional in two triangles" be a classification criterion for similar triangles?*

Y: *Yes, this is the theorem. And we sort similar- triangles by it.*

I: *One student defined similar triangles as follows: "Two triangles are similar when all of their corresponding sides have lengths of the same ratio." Can you accept it as a correct definition?*

Y: *Yes, I can accept it as a correct definition, because in this definition, the essence of the concept is clear.*

I: *Does the [aforementioned] statement equivalent to the statement "two angles of one triangle are equal to two angles of the other triangle"?*

Y: *Yes, because from one theorem we can conclude the other theorem.*

I: *Why one theorem you accepted as definition and the other you didn’t accept?*

Y: *Because of the essence of the concept. One gave us the essence and the other not.*

I:*lets go to the congruent triangles concept, “two triangles, △ABC and △A’B’C’ are congruent if all the three side are equal” are you accepted it as a definition for congruent triangles.*

Y: *Yes, I can accept it.*

I: *What about “two triangles, △ABC and △A’B’C’ are congruent if two angles and the inscribed sides are equal” as formal definition?*

Y: *No, I can't accept it as definition.*

I: *Why?*

Y: *Another time it didn’t give us the essence of the concept.*

I: *Although it is a theorem for congruency?*

Y: *No, it can’t be a formal definition.*

I: *but it is written like definition?*

Y: *Definition have to give us insight about the concept.*

I: *What are the roles of the congruent triangles' theorems.*

Y: *To prove that the triangles are congruent triangles.*

I:*So it can classify congruent triangles?*

Y: *Yes.*

I: *And couldn’t be definitions?*

Y: *No.*

I:*Why?*

Y: *There is only two definitions, one includes all the attributes and the other gives the equal sides of the triangles and twice give the meaning of concept.*

I: *I want to tell you that the definition must contain necessary and sufficient attributes to sort examples and non-examples.* *Could you change your answer about the congruent triangles' theorems?*

Y: *What to change?*

I: *If they could be definitions?*

Y:*. No, only side, side, side could be a definition.*

I: *O.K thank you for your answers.*

Y: *Your welcome*.

Interview 2: Samir

Interviewer: *Hi, I just ask you to define similar triangles.*

Samir: *O.K, similar triangles are couple of triangles which have equal angles and the sides have same thing…no the sides are proportional.*

I: *This can used in criterion to sort similar triangles?*

S: *Yes, we can.*

I: *In the questionnaire, you claimed that only Sami’s definition […] is right.*

S: *Yes.*

I: *Can we use Sami's to prove similar triangles?*

S: *Yes, because it give only the similar-triangles.*

I: *So, it could be a classification criterion for similar triangles?*

S: *Yes.*

I: *why?*

S: *It’s the known theorem.*

I: *why it couldn’t be a definition for similar-triangles theorem?*

S: *Because there is one definition, Rami gave a definition and Sami gave a theorem, and there is deference between them.*

I: *What is the deference between them.*

S: *In the text-books Rami's is accepted as definition and Sami's as a theorem.*

I: *Do you think that for one concept there is only one definition?*

S: *Yes, I do.*

Interviewer: *did you accept the statement “two triangles, △ABC and △A’B’C’ are congruent if all the three side are equal” as a definition for congruent triangles.*

Samir: *No, I can't accept.*

I: *Although it is a theorem for congruency?*

S: *Yes. Because there is a difference between definition and theorem.*

I: *what is the difference?*

S: *Sami's is the definition and there is only one known and accepted definition, the other is theorem which one have to prove.*

I: *What are the roles of the congruent triangles' theorems.*

S: *To identify congruent triangles from non-congruent triangles …and help us to prove that two triangles are congruent.*

I: *So it is a base to decide whether two triangles are congruent or not congruent?*

S: *Yes.*

I: *And couldn’t be definitions?*

S: *No, it couldn't be a definition.*

I: *I want to tell you that for one concept could be more than one definition. the definition must contain necessary and sufficient attributes and some of the roles of definitions are to sort examples and non-examples of the concept and to be base for proofs.*

S: *…..*

I: *Could you change your answer about the congruent triangles' theorems?*

S: *What to change?*

I: *If they could be definitions?*

S: *I think yes; they can be definitions.*

I: *And what about similar triangles' theorems?*

S: *although it difficult for me to accept the first theorem (angle, angle), but these theorems could be definitions for similar triangles concept.*

I: *Why it is difficult for you to accept the first theorem as definition?*

S: *Because it didn’t give the meaning of the concept about proportion.*

I*: Thank you Samir for your answers.*

S*: Your welcome.*