**Tracking the growth of mathematical ideas in a professional development program for kindergarten teachers**

**ABSTRACT**

Professional development (PD) programs can provide a basis for mathematical knowledge growth. This study was conducted during a PD program for kindergarten teachers that used argumentation discourse within a whole-class discussion to promote geometrical knowledge. To follow the evolution of the teachers’ mathematical ideas, we adopted a Documenting Collective Activity framework. This approach allowed us to identify normative ways of reasoning that functioned as-if-shared in the teachers’ collective argument discourse during the PD program.

**1. Introduction**

Tracing the emergence, travel, and shifts of learners’ ideas in classroom communities, is a challenge that has yet to be achieved (Saxe et al., 2009). In recent years, studies have emphasized the importance of considering mathematics classrooms as learning communities rather than as individuals (e.g. see Hodge, 2008). Thus, a growing number of studies have explored the collective mathematical activity of learning communities (e.g. see Rasmussen et al., 2015). The collective activity of mathematics learning refers to the normative ways of reasoning that develop as learners work together to explain their thinking, present their ideas, and solve problems (Cobb et al., 2001).

While the studies noted above explore the collective activity of mathematical learning, there remains a need for research that focuses specifically on kindergarten teachers. PD programs can provide a basis for the growth in mathematical knowledge that is necessary for teaching (Tirosh et al., 2011). The main goal of this study is to identify and understand the learning processes in a PD program for kindergarten teachers by tracking the growth of their mathematical ideas. We investigated kindergarten teachers’ argumentation discourse during whole-class discussions about three-dimensional shapes.

**2. Theoretical and methodological framework**

***2.1 Knowledge development in the classroom***

Over the last few years, research has increasingly focused, theoretically and methodologically,

on knowledge development during the learning process. Cobb and Bauersfeld (1996) provided a basis for this development by presenting research on mathematical learning from a psychological and sociocultural perspective, and proposed a theoretical-methodological framework to study the mathematics classroom. Nonetheless, learning mathematics via social interaction is an integral part of the socio-mathematical approach to learning.

The socio-mathematical approach, which is rooted in constructivist theory, argue that the development of mathematical reasoning from social and psychological perspectives should be the focus of research. From these perspectives, students learn the classroom’s prevailing socio-mathematical norms together with other students (Cobb & Yackel, 1996). Cobb et al. (2001) expanded on the significance of the development of mathematical knowledge in the classroom. Their main aim was to study this as it is expressed in the growing collective activity of the classroom community. This social perspective stresses the mathematical practices that become normative ways of reasoning within the classroom community, constructed through ongoing interaction among students. These practices are taken-as-shared in the classroom community. The psychological perspective relates to the students’ qualitative development in terms of their ability to reach logical conclusions, i.e. to reorganize their activity and the world in which they function. This perspective emphasizes how different students in the classroom community participate in these activities. Meanings or mathematical activities are defined in the classroom as either taken-as-shared or not taken-as-shared, since different students think in different ways.

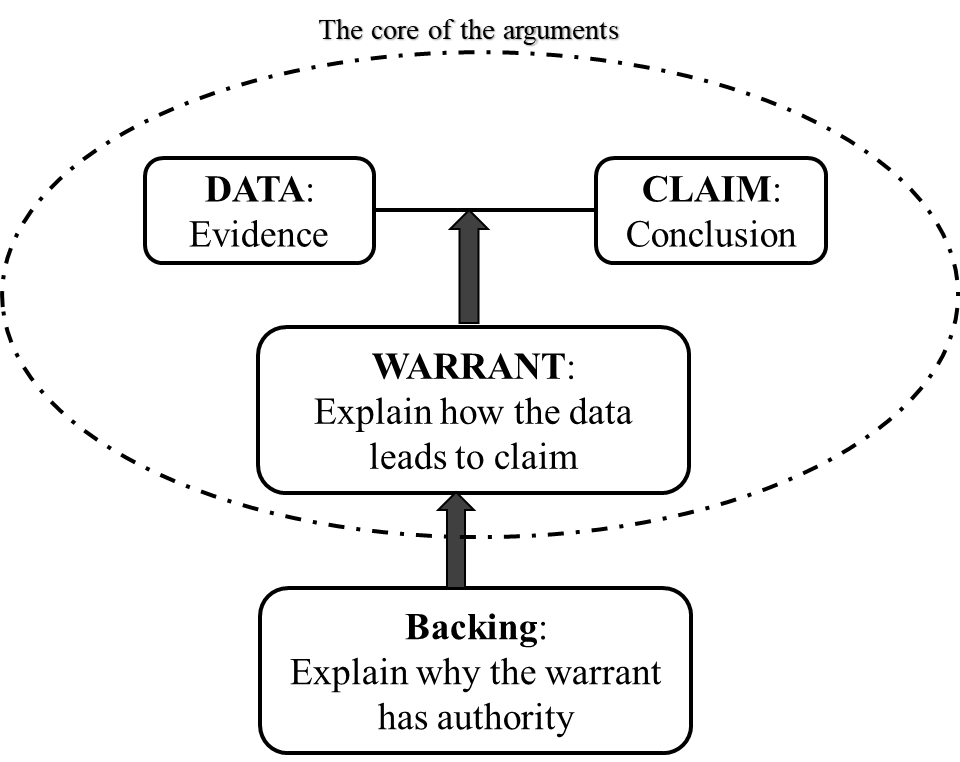
Subsequently, we describe the theoretical-methodological framework we used to analyze the discussions that took place in the whole-classroom discussions. This framework is the focus of this paper, and enables us to clarify how the students’ justifications or warrants actually influence classroom practice.

***2.2 Documenting Collective Activity (DCA)***

Collective activity is a social phenomenon that addresses the constitution of ideas through patterns of interaction. It allows mathematical ideas to become normative ways of reasoning that are established in the classroom community through student interaction. As the students present their ideas, and explain their thinking behind them, the ideas take shape.

A mathematical idea or way of reasoning is considered normative when there is empirical evidence that it functions in the classroom as if it had been shared via an argumentation process that reaches logical conclusions using language, tools, symbols, and gestures. As applied to mathematical ideas, the term “function as if shared” means that mathematical ideas and ways of reasoning behave within the classroom discourse “as if” everyone in the classroom community had been a factor in their creation. The concept “shared” is introduced to create a closer connection to the empirical approach that establishes when certain ways of reasoning become normative. The empirical evidence that establishes the ideas, behaves as per Toulmin’s model of argument (1969) (Rasmussen & Stephan, 2008; Stephan & Rasmussen, 2002), assuming that there are differences in how different individuals in the classroom perceive mathematical ideas and how conclusions are reached.

Toulmin (1969) created a model (see Fig.1) to depict the structure and function of the argument, called the core of the argument, which is composed of at least three parts: the grounds, the claim, and the warrant.



*Fig. 1 Toulmin’s model of argument*

In the argument, the speaker makes a claim and presents evidence or data to support the claim. The data is usually composed of facts or procedures that lead to a conclusion. If a student does not understand how the data leads to the conclusion, she can ask the speaker for clarification. To strengthen the claim, the speaker often provides a clarification that serves as a warrant, connecting the data to the claim. Nevertheless, it is not unusual for rebuttals or qualifiers to be raised after the claim is made, the data is given, or the warrant is provided. The rebuttals and qualifiers help advance the claim.

Sometimes a listener may understand why the data supports the conclusion, but still disagree with the warrant the speaker produced, i.e. the listener may call into question the authority or credibility of the warrant such that the speaker must provide a rebuttal to explain why the warrant and the core of the argument are valid. If a student understands why the data supports the conclusion but disagrees with the contents of the warrant advanced by the speaker or with the claim, she can make rebuttals or offer counter-arguments that testify to her disagreement. When this type of objection is made, a qualifier is often provided as a way to offer specific conditions under which the claim is true. Finally, the argument may also include backing, which demonstrates why the warrant has the authority to support the data-claim pair.

*Stages of DCA*

The first stage begins by employing Toulmin’s model to create a sequence of argumentation schema for the discussions taking place in the whole-class discussion. From these, we constructed an argument log for these discussions. The following stage shows a review of the argument log to identify which mathematical ideas became accepted as part of the normative ways of reasoning, i.e. the normative ways of justification in the classroom community and function as if shared.

Stephen and Rasmussen (2008) defined two criteria for determining when a classroom community’s ways of justification or mathematical ideas become normative:

* Criterion 1 – Drop off: This occurs when the backings and/or warrants for an argument cease to be mentioned in the students’ explanations (they are hinted at, instead of being explicitly asserted, no community member challenges the argument, and/or if a student objects to the argument, the challenge is refuted). We conclude that within the classroom community the mathematical idea at the heart of the argument has become obvious, i.e. this criterion is fulfilled when the same conclusion is discussed in more than one lecture or more than once in the same lecture, and in subsequent events, the backings or warrants are omitted.
* Criterion 2 –Shift of position: This occurs when one of the elements of Toulmin’s model of argument (1969), the warrant, claim, data, or backing, changes its role during the lecture or afterward, i.e. every time one of these components changes its role in the argument and no one objects (or, if someone does object, the objection is refuted), we may conclude that the mathematical idea has become normative in the class and functions as if shared. For example, when a student employs a claim that had been previously justified as a component (data, warrant, or backing) in his current argument, we can almost certainly conclude that the mathematical idea expressed in the claim has become a normative way of justification in the classroom.
* To these, Cole et al. (2011) added a third criterion: repeated use. This occurs when the same data or warrants are employed for different arguments, i.e. when a certain idea is recycled either as data or as a warrant.

**3. Methodology**

***3.1 Research question***

The main goal of this study is to identify and understand the learning processes in a PD program for kindergarten teachers by tracking the growth of their mathematical ideas. Thus, a methodology is required to analyze data from the whole-class community. The overall goal can be expressed by the following research question: How can we describe and characterize the learning process of teachers participating in a PD program?

***3.1 Research setting***

The data reported in this paper were collected as part of a larger study of a PD program for kindergarten, first grade, and second grade teachers about two-dimensional (2D) and three-dimensional (3D) shapes in geometry. Due to space limitations, we focus here on the kindergarten teachers' PD program, which included 20 kindergarten teachers and met for 10, three-hour PD sessions. Study data sources included video recordings of each class session from a single camera that focused on whoever was speaking during whole-class discussions.

***3.2 Method***

Twenty kindergarten teachers employed in the State Education track, who had participated in the PD program, took part in this study. The program focused on increasing the participants’ knowledge of shapes and objects, and how to teach them. The basic aims of the program were as follows: (a) to improve the professionalism of kindergarten teachers in terms of their content-related and pedagogical knowledge of kindergarten geometry; (b) to develop pedagogy to improve geometry teaching in general, and the teaching of shapes and forms in particular; (c) to develop activities based on content-related and pedagogical knowledge in kindergarten geometry pedagogy. These fundamental aims led to certain specific goals: (1) to develop familiarity with the place of geometry within the pre-elementary school teaching program; (2) to develop familiarity with the principles guiding the development of geometric thinking in kindergarten (the Van Hiele Model); (3) to expose teachers to problems that kindergarteners encounter in different geometry subjects and help the teachers understand them; (4) to expose teachers to actual teaching aids that represent and demonstrate 2D and 3D shapes and forms.

The pyramid was chosen as the paradigmatic object that would be used to analyze the learning process as reflected in the whole-class discussions that took place within the program. Below is our analysis of one session—the ninth of ten—that addressed the topic of the pyramid shape.

**4. Analysis of Session 9—Findings**

In this session of the program, the focus was on correctly identifying which of a group of various shapes were pyramids, and providing justifications for these choices. Following this activity, the facilitator conducted a discussion with the participants about the selected shapes. During the discussion, the participants determined which characteristics had helped them distinguish the pyramids from the other geometric shapes.

The session was divided into three sections: the first involved group work, the second contained multiple whole-class discussions, while in final section the group watched and analyzed a short film that showed an interview with a kindergarten pupil about identifying pyramids.

We segmented the whole-class discussion section into episodes. Table 1 summarizes the chronological order of the episodes. We then analyzed the whole-class discussion episodes using DCA.

Table 1 Lesson 9 — Summary of Analysis

|  |  |
| --- | --- |
| Episode number | Title |
| Ep. 1 | Defining Pyramids and Polyhedrons |
| Ep. 2 | Properties of the Square Pyramid |
| Ep. 3 | Properties of the Hexagonal Pyramid |
| Ep. 4 | Properties of the Triangular Pyramid |
| Ep. 5 | Properties of the Prism |
| Ep. 6 | Properties of the Truncated Pyramid |
| Ep. 7 | The Cylinder |
| Ep. 8 | Properties of the Pentagonal Pyramid |
| Ep. 9 | The Rectangular Pyramid |
| Ep. 10 | The Inverted Pyramid (Apex Below & Base Above) |
| Ep. 11 | Rectangular Prisms |
| Ep. 12 | Truncated Pyramids |
| Ep. 13 | Square Pyramids |
| Ep. 14 | גוף מדרגות |

***4.1 Whole-class discussion***

The session began with a whole-class discussion (Episodes 1–4, Utterances 1–32) about polyhedrons and specifically how to define pyramids.

*Episode 1: Defining Polyhedrons and Pyramids*

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| --- | --- | --- |
| **Line** | **Speaker** | **Utterance** |
| 5 | Facilitator | OK, now that all the groups have finished the activity, let’s start the second part.  As I remarked earlier, our topic for today is the pyramid. Let me remind you what we’ve learned about shapes. We studied shapes that are not polyhedrons.  Who can remind me what the definition of a polyhedron is, and which polyhedrons we have not studied yet? |
| 6 | Teacher 1 | As I recall, the polyhedron is a shape composed of polygons. |
| 7 | Facilitator | Correct.  So what shapes have we learned so far? |
| 8 | Teacher 1 | We studied the objects that are not polyhedrons. These objects are not composed of polygons. We learned about spheres, cones, and cylinders. None of those are polyhedrons because they are not composed of segments. |
| 12a | Facilitator | Great. Now, this leads us right to the topic we are going to study in-depth today, one of the polyhedrons called the pyramid. Together, we will discuss and investigate what a pyramid is. What characterizes it? How can we teach it to kindergarteners? We will also decide which of the objects that you constructed are pyramids.  I have placed the objects you created on the table. I will present the objects one at a time, and I will ask you to judge each object individually and decide whether or not it is a pyramid. |
| 12b | Facilitator | However, before I do that, what do you know about the pyramid? What do you think a pyramid is? What are the properties or elements of the pyramid? |
| 13 | Teacher 2 | The pyramid has a square base and four polygons, which are essentially four triangles. |
| 14 | Teacher 3 | The base is square, not a square, rather squared, and the pyramid’s lateral faces or sides are triangles, and there are only four triangles. |
| 15 | Facilitator | What do you have to say about what has been said about the pyramid? |
| 16 | Teacher 4 | Correct. A pyramid has four triangles. |
| 17 | Facilitator | Do you have anything to add about the pyramid? |
| 18 | Teacher 5 | I think the pyramid has three triangles. |
| 19 | Facilitator | [Addressing the class] What do you think? Does the pyramid have three or four triangles? |
| 20 | Teacher 6 | Four, not three! |
| 21 | Teacher 7 | Maybe the pyramid has six triangles? |
| 22 | Facilitator | What do you all think? Does the pyramid have three, four, or six triangles? |
| 23 | Teacher 3 | The pyramid has four triangles. |
| 24 | Teacher 7 | The pyramid has four polygons that are triangles and its base is a square polygon. |
| 25 | Teacher 8 | The pyramid has a square base and each one of the edges that help construct the base is in fact the base of one of the triangles that compose the pyramid. |

The discussion began with a brief introduction of the pyramid, and an immediate opening question regarding how to define the concept of a polyhedron (5) and which objects had been studied so far (7). The first claim was made by Teacher 1 (6, 8) in the form of a response to the facilitator’s question. In terms of Toulman’s Model, Teacher 1’s claim can be broken down as follows:

Argument 1: (5-9) Defining the polyhedron

Claim 1: A polyhedron is an object that is composed of polygons. (6)

Data 1: She remembers.

Warrant 1: We learned a bit about the cube and the box—both of these are composed of polygons. (9)

Later, the second argument about non-polyhedral objects was made by the same teacher, Teacher 1.

Argument 2: (7-11) Defining a Non-Polyhedral Form

Claim 2: Non-polyhedral objects are not composed of polygons. (8)

Data 2: We studied spheres, cones, and cylinders. None of these are polyhedral. (8)

Warrant 1: Because they are not composed of polygons (8).

The facilitator continued by asking what the kindergarten teachers already knew about the pyramid. In response to the facilitator’s main question about the elements and properties of the pyramid (12b), a new argument was raised (Argument 3) about the two elements that comprise the definition of a pyramid: the first pertaining to the shape of the pyramid’s base, and the second to the triangular shape of the pyramid’s lateral faces. Teacher 2 was the first to raise the argument about these two elements (13). Following Teacher 2’s remarks, some teachers objected (Objection 2a and 2b) either to the first element (Teacher 3, Column 14; Teacher 5, Column 18) or to the second element (Teacher 6, Column 18, Teacher 7, Column 21); some teachers supported the claim about the first element (Teacher 8, Column 25); some the second element (Teachers 3, 4, 6, respectively, Columns, 14, 16, 20), and some supported both claims (Teacher 7, Column 24). Ultimately, Teacher 8 summed up the discussion. Teacher 8 was the only one who offered a warrant for the connection between the two elements (25). In terms of the Toulman model:

Argument 3: (13-25) Elements of the Pyramid

Claim 3: The pyramid has a square base and four polygons that are in fact 4 triangles. (13)

Data: Based on Memory

Warrant 3: Each one of the edges that form the base is actually the base of one of the triangles that form the pyramid. (25)

Objection 3a: The base is squared, not a square, but rather squared. (14)

Objection 3b: The pyramid has three triangles. (18)

Objection 3c: Four. Not three! (20)

Objection 3d: The pyramid has six triangles. (21)

*Episode 4: Hexagonal Pyramid Properties*

The facilitator presented the following shape (a hexagonal pyramid) and first asked the group that had constructed it to describe how they did so. Following this, she asked them to decide whether or not the object was a pyramid.

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| --- | --- | --- |
| **Line Number** | **Speaker** | **Utterance** |
| 37b | Facilitator | Let’s begin studying the next form. However, before doing so, I will address my first request to the group that constructed this form. Please explain the stages of your construction of this object. |
| 38 | Teacher 7 | I’ll explain. That’s our group. |
| 39 | Facilitator | Go ahead. |
| 40 | Teacher 7 | First of all, this is a long object. Or, in other words, the triangles that form it possess two sides that are both longer than the third one. |
| 41 | Facilitator | Correct. But what did you do in this case? |
| 42 | Teacher 7 | First of all, we constructed the triangles, which are in effect six triangles, like the first group, but we used the thinnest sticks available, and then we connected all of them together at the common vertex above and also below so that every two neighboring triangles that comprised the object also converged at a common vertex on the base of the triangles. As a result of these convergences, there were two sticks between every two neighboring triangles. However, since we used thin sticks, we connected each of these two neighboring segments with Scotch tape. Therefore, we did not have to remove sticks as the first group did. Of course, this process took time since we had to strategize the construction process. |
| 43 | Facilitator | Wonderful! So the second group began creating long triangles—six triangles with equal sides, which are also congruent in this case, and then they joined all the vertices (since every triangle’s vertex, in fact, connects the two long segments of the isosceles triangle) into a common vertex. In addition, they also connected every two neighboring triangles, with a common segment and with their common neighboring vertex in the base. In this way, a base was formed. The question remaining is, Is this form a pyramid? |
| 44 | Teacher 7 | Before I answer, I’ll explain a few things. At first, I was unsure whether a pyramid had six, four, or three triangles. When you asked us what the properties of a pyramid were, I answered, “Maybe a pyramid has six triangles!” However, in the end, I decided that this wasn’t a pyramid, since most of the kindergarten teachers at the beginning of the meeting said that a pyramid had three or four triangles. However, now I’m willing to say that this is definitely a pyramid because of the explanation we gave about the previous pyramid. I understand that a pyramid has triangles that are mmmmmm… the faces of the pyramid. |
| 45 | Facilitator | Fine. Do you have anything else to add? |
| 46 | Teacher 15 | I’ll continue since the two of us were in the same group. I’ll add that these triangles have one common point, which is the top of the pyramid—and the base is also formed by the convergence of these triangles. |
| 47 | Facilitator | Does anyone have a different opinion about this shape? |
| 48 | Teachers | We agree. This is a pyramid. |
| 49 | Facilitator | Wonderful! Quite correct. Let me summarize. This object is a pyramid, since its lateral faces are triangles, they all converge at one apex, which is the top of the pyramid, and the base is polygonal. Therefore, this is a pyramid. |
| 50 | Teacher 2 | So it doesn’t matter what shape the base is. Can it be any polygon? I noticed that the base of this pyramid is a hexagon. |
| 51a | Facilitator | Correct. And this is one of the pyramid’s important properties. The base can be any polygon. |

After the facilitator accepted the explanation given by the two teachers about their strategy for constructing this form (42-43), she asked them to decide whether the object they had created was a pyramid or not (43). Responding to her question, Teacher 7 raised a new argument, claiming that the object was a pyramid and providing a warrant for her assertion (44). Adding to Teacher 7’s remarks, Teacher 15 provided backing that explains the warrant for this argument (46).

Argument 5: (Column 37-51a) The Elements of a Hexagonal Pyramid

Claim 5: This is definitely a pyramid (44).

Data 5: Hexagonal PyramidA picture containing scale, deviceDescription automatically generated

Warrant 5: The pyramid possesses triangles that comprise its lateral faces (44)

Backing 5: The triangles have one common point, which is the top of the pyramid, and the base is formed by connecting these triangles.

Argument 5 is diametrically opposed to Argument 2 (Columns 13-25), since in Argument 3 everyone agreed that a pyramid has four polygons which are triangles, a polygonal base that is square (24), and wherein each one of the base’s edges is part of one of the triangles that comprise the pyramid (25).

We should note that there were several kindergarten teachers who agreed with everything that Teachers 7 and 15 said in Argument 5 (48). This broad consensus is a sign of normative agreement that we believe strengthens the teachers’ remarks in the context of Argument 5. Therefore, we can conclude that this provides indirect first-hand evidence of the notion that over the course of the discussion, the idea that “*the elements of the pyramid: triangular faces, a base and a top that are formed by the convergence of these triangles*” became an idea that functioned as-if-shared.

In light of the facilitator’s summary about the hexagonal pyramid form (49), Teacher 2 raised Argument 6 (50), which relates to the shape of the pyramid’s base.

Argument 6: (Column 50) The base of every pyramid is a polygon.

Claim 6: Any polygon can be the base.

Data 6: The base of this pyramid is a hexagon.

Warrant 6: It doesn’t matter what [polygon] the base is.

The base of the pyramid, which appeared as a backing in Argument 5 (46), is mentioned again by Teacher 2, but this time, it is repeated as a component of the warrant for Argument 6 (50). According to the second criterion of the DCA methodology, this form of recall is defined as a *shift of position*. Therefore, we may conclude that during the course of the discussion, the idea that “*the base of the pyramid is a polygon*” became an idea that functioned as-if-shared.

*Episode 4: Properties of the Triangular Pyramid*

The facilitator’s next step was to examine the properties of a triangular pyramid. The facilitator presented triangular pyramid shapes that the kindergarten teachers had constructed and facilitated the following discussion:

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| --- | --- | --- |
| **Line Number** | **Speaker** | **Utterance** |
| 51b | Facilitator | Let’s turn our attention to the next form. Is it a pyramid? |
| 52 | Teacher 8 | Yes, it is a pyramid because this pyramid has a top, which is the point at which all the triangles surrounding [sic] the pyramid converge. And the base is polygonal. |
| 53 | Facilitator | Correct. This is the top of the pyramid, and these are the triangles that are the pyramid’s faces, and that is the base. [She points to each of these features on the actual object she is holding.]  My request is: Please look at the base . |
| 54 | Teacher 9 | The base is the triangle on the bottom. |
| 55 | Facilitator | [Rotates the object] Okay. And now where is the base? A picture containing triangle, music, tripodDescription automatically generated |
| 56 | Teacher 9 | Maybe it’s the one on this side [pointing to the right side], or….maybe in the back. |
| 57 | Facilitator | [Rotating the object again] Okay, and now where exactly is the base? Which triangle is the base? A picture containing triangle, music, tripodDescription automatically generated |
| 58 | Teachers | [Looking at it] |
| 59 | Teacher 10 | I have something to say. |
| 60 | Facilitator | Please do. |
| 61 | Teacher 10 | Probably the base of this pyramid is also a triangle. We can assume that every one of the pyramid’s triangles could function as the pyramid’s base. |
| 62 | Facilitator | What do you think? |
| 63 | Teacher 8 | Correct. This pyramid fulfills the properties we have learned. Therefore, we can make this assumption. |
| 64 | Facilitator | Does anyone have a different opinion? |
| 65 | Kindergarten Teachers | No. That makes sense. |
| 66a | Facilitator | Teacher 10’s remarks are correct. This is an important conclusion to reach about this pyramid. This pyramid is composed of congruent triangles [equilateral triangles]. Therefore each and every side can also be treated as the base of the pyramid. It depends on the pyramid.  Look at the chart that I drew while you were answering my questions. Note that among the pyramids you constructed, there is a connection between the number of triangles [in each pyramid] and the type of base.  I will summarize the important, defining characteristics of a pyramid that we have discovered so far:  A pyramid is a shape composed of a shell, all faces of which are triangular and share an apex and a polygonal base.  We can also define a pyramid as a form that is constructed from the polygon referred to as the pyramid’s base and a point outside the plane, which is referred to as the pyramid’s top, and from all the triangles that are formed by the polygon’s point and segments, which are the pyramid’s faces. |

After the facilitator asked the teachers to decide whether the object shown was a pyramid or not (51b), Teacher 8 raised Argument 7 regarding the three components of the pyramidal form: the top (apex), the faces (lateral sides), and the polygonal base (52).

Argument 7: (51b-52) Components of the Triangular Pyramid

Claim 7: This is a pyramid. (52)

Data 7: Triangular Pyramid (A picture containing triangle, music, tripodDescription automatically generated)

Warrant 7: Because this pyramid has a point at the top, where all the triangles forming the pyramid meet, and there is also a polygonal base. (52)

The contents of the warrant in Argument 7, which relate to defining the pyramidal shape by adducing the components of a pyramid—the apex, the shape of the triangles, and the polygonal base (52)— were noted previously in Column 44 with regard to the triangles that form the faces of the pyramid (as Warrant 5 in Argument 5) and in Column 46 with regard to the composition of the pyramid’s apex and polygonal base (as Backing 5 in Argument 5). According to the second criterion, this form of recall concerning the definition of the pyramid is a *shift of position*. Therefore, we may conclude that during the course of the discussion, the idea that a pyramid is ***“composed of an apex, a polygon base, and triangular faces”*** became an idea that functions as-if-shared.

Following this, the facilitator attempts to change the object’s name right in front of the teachers, to sow confusion regarding which of the polygons was the base of the triangular pyramid. She did this by asking the teachers to identify where the base was each time she rotated the object (53, 55, 57). In response to these attempts, the following argument was raised:

Argument 8: (61-63) A polygon is the triangular pyramid’s base.

Claim 8: Every one of the triangles that comprises this pyramid can be its base. (61)

Data 8: Triangular Pyramid (A picture containing triangle, music, tripodDescription automatically generated)

Warrant 8: This pyramid’s base is also a triangle. (63)

Several kindergarten teachers agreed with everything the teachers said in this discussion (51-66). This broad consensus is a sign of normative agreement that we believe strengthens the teachers’ utterances in the contexts of Argument 7 and Argument 8. Therefore, we may conclude that this provides evidence of the notion that over the course of the discussion ***“each of the triangular pyramid’s triangles can be the pyramid’s polygonal base”*** became an idea that functioned as-if-shared.

*Episode 5: The Properties of the Prism*

The facilitator presented the prism shape that had been created by one of the groups (during the first part of this session). She then asked the teachers to determine whether this object was a pyramid or not and to justify their conclusions.

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| --- | --- | --- |
| **Line Number** | **Speaker** | **Utterance** |
| 66b | Facilitator | We will turn our attention to the next form. Is it a pyramid? |
| 67 | Teacher 11 | No. It is not a pyramid. |
| 68 | Facilitator | How do you know? Please explain yourself. |
| 69 | Teacher 11 | Because the sides are not just triangular. There are also rectangles. |
| 70 | Facilitator | Kindergarten teachers, what do you think? |
| 71 | Kindergarten Teachers | Ahhh, correct. She is right. |
| 72a | Facilitator | Okay. You all agree. All the pyramid’s faces have to be triangles. Therefore, this is not a pyramid.  Therefore, it is only necessary to discover one critical attribute—that is to say, one property that every pyramid must possess, which cannot be relinquished…in order to exclude it from the collection of different pyramid types. And in this form, the critical property is missing since all the sides must be triangular. Conceptually speaking, the sides are called the pyramid’s faces. |

This discussion is brief and all the teachers agree that this object is not a pyramid. Teacher 11 is the teacher participating in this discussion. She raises the claim that this shape should be eliminated from the collection of pyramidal shapes (67) and provides a warrant for her claim (69).

Argument 9: (66-72) This object is a pyramid.

Claim 9: Not a pyramid (67)

Data 9: The Prism (IconDescription automatically generated)

Warrant 9: The sides are not solely triangular. There are also rectangles. (69)

After Argument 9 was made, several kindergarten teachers who agreed with everything that the teachers who participated in this discussion said (Columns 66b-72a). This broad consensus is a sign of normative agreement that we believe strengthens the teachers’ utterances in the context of Argument 9. Therefore, we may conclude that this provides indirect evidence of the notion that over the course of the discussion ***“if the shape has sides that are not triangular then the shape is not a pyramid”*** became an idea that functions as-if-shared.

*Episode 8: The Properties of the Pentagonal Pyramid*

The discussion concerning the shape (the pentagonal pyramid) follows below:

|  |  |  |
| --- | --- | --- |
| **Line Number** | **Speaker** | **Utterance** |
| 85b | Facilitator | The next object. Is this object a pyramid? |
| 86 | Teacher 14 | Definitely. We already built this. |
| 87 | Facilitator | You built the pyramid with a hexagonal base.  Take a look at the base. What polygon is this?  Please tell us why this is a pyramid too. |
| 88 | Teacher 14 | One, two, three, four, five. It’s a pentagon.  Correct. So this is a pyramid with a pentagonal base because this is a polyhedron composed of triangles that converge on a common vertex, which has a polygonal base that is pentagonal. |
| 89 | Teacher 2 | You cited the definition itself. Just say that the object fulfills the parameters of the pyramid. |

After the facilitator requested that the teachers decide the identity of the shape that she presented via computer (85b), Teacher 14 deliberated and claimed that this shape had been constructed by one of the groups and that they had identified it as a pyramid (86). However, after examining the polygonal base and the other properties of the shape, she raised Argument 12 and provided a warrant for it. In the discussion that followed about the shape, Teacher 2 provided a summary of the reasons why she believed it was a pyramid (89).

Argument 12: (88-89) This is a pentagonal pyramid.

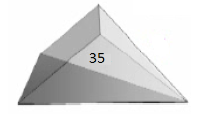
Claim 12: This pyramid has a pentagonal base. (88)

Date 12: The form presented (a pentagonal pyramid).

Warrant 12 Because this is a polyhedron composed of triangles that converge at the apex and has a polygonal base that is pentagonal. (88)

Backing 12 This form fulfills the parameters of a pyramid. (89)

*Episode 9: A Rectangular Pyramid (Column 90b - Column 97)*

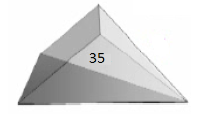
The facilitator continues by presenting another form to stimulate the teachers’ thinking.

|  |  |  |
| --- | --- | --- |
| **Line Number** | **Speaker** | **Utterance** |
| 90b | Facilitator | Next one. Is Object Number 35 a pyramid? |
| 91 | Teacher 8 | No. Not all the faces are triangles. There is a square on the left side. |
| 92 | Facilitator | What do you think? |
| 93 | Teacher 2 | Maybe I am wrong, but you can rotate it. This is a pyramid that is lying on one of its sides, not on its base. |
| 94 | Facilitator | To return to Teacher 8…. Please take a look at the object, and think again about whether or not it is a pyramid. |
| 95 | Teacher 8 | No. It still seems to me that this object is similar to the object that we constructed and that we agreed was not a pyramid. |
| 96 | Teacher 2 | This object does, indeed, fulfill the parameters of the pyramid. We can construct it and prove it to you. |
| 97 | Facilitator | Teacher 8, I respect your opinion. You may construct it and examine it at home, and let us know if you have changed your mind.  Teacher 2 was right. To sum it up, you need to take a very good look at the object. Maybe it’s lying on one of its sides. Whenever you look at the object, look for the critical attributes. For instance, in Object 35, you need to look for the number of triangles that comprise the object. After this, [ask] if they have one common apex, and finally, you need to examine the polygonal base, which is a rectangle, and is parallel to the common apex. And that is that. |

In response to the facilitator’s question about the object on display (90b), Teacher 2 raises Argument 13, which claims that the object is not a pyramid, and also includes a warrant (91). However, Teacher 8 opposes Argument 13, claiming that the object is indeed a pyramid (93). Following this, Teacher 8 responds, disagreeing with Teacher 2’s claim in Argument 14. She still bases herself on the object’s external appearance (95). Here are the two opposing arguments:

Argument 13: (91-95) This is a rectangular pyramid.

Claim 13: Not a pyramid (91)

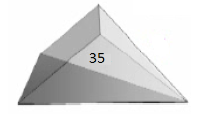
Data 13: The object on display (a rectangular pyramid)

Warrant 13 Not all the sides are triangles. There is a square on the left side. (91)

Backing 13: This object is similar to the object that we constructed and that we agreed was not a pyramid. (95)

Argument 14: Opposing Argument 13

Claim 14: This pyramid is lying on one of its sides, not on its base. (93)

Data 14: The object on display (a rectangular pyramid)

Warrant 14: This object does, indeed, fulfill the parameters of the pyramid. (96)

Backing 14: We can construct it and prove it to you.

Opposition 14: No. It still seems to me that this object is similar to the object that we constructed and that we agreed was not a pyramid. (95)

The individual difference between Teacher 8 and Teacher 2 that emerged through their opposing arguments (13 and 14) stresses the crucial theoretical concept of an idea or way of thinking functioning as-if-shared. In our analysis of Argument 10, we established that the idea of “checking the definition” functioned as-if-shared. This does not mean that all the teachers had identical perceptions, as can be clearly seen in the case of Teacher 8.

In surveying the results of our examination of all the objects on display in light of the pyramid’s parameters, we realized that the phrase ***“the parameters are not met”*** appeared in Backing 12 and in Argument 12 (89) and then appeared again in Argument 14 and also as a warrant (96). According to the shift of position criterion we may conclude that the phrase ***“the object fulfills the parameters of the pyramid”*** functions as an idea as-if-shared within this classroom community.

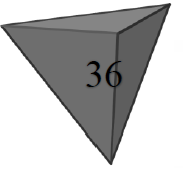
Episode 10: An Inverted Pyramid – Apex Below and Base Above

|  |  |  |
| --- | --- | --- |
| **Utterance** | **Speaker** | **Line Number** |
| Next object. Is Object Number 36 a pyramid? | Facilitator | 98 |
| Yes. It is an inverted pyramid that presents in the opposite direction of the one we’re used to seeing. We made one like this. And there is also the base, which is formed by the convergence of those triangles, and, in particular, as a result of the triangles’ bases connecting. Every two bases of two adjacent triangles converge on one apex which is the top of the pyramid. | Teacher 7 | 99 |
| Are all three fundamental properties present in this object? | Facilitator | 100 |
| Yes. The pyramid meets all the parameters. | Teacher 7 | 101 |

In response to the facilitator’s question about the object being displayed (98), Teacher 7 raises Argument 15 which claims that the object is not a pyramid and provides a warrant (99) and backing (101).

Argument 15: (99-101) This is an Inverted Pyramid.

Claim 15: This pyramid is inverted. (99)

Data 15: The object on display 

Warrant 15: Yes. It is an inverted pyramid that presents in the opposite direction of the one we’re used to seeing. We made one like this. And there is also the base which is formed by the convergence of those triangles, and, in particular, as a result of the triangles’ bases connecting. Every two bases of two adjacent triangles converges on one apex which is the top of the pyramid. (99)

Backing 15: The pyramid meets the parameters [of a pyramid]. (101)

The use of the phrase ***“the pyramid meets the parameters”*** (101) is repeated during the course of this session’s discussion as backing in Argument 15. This is further proof that the phrase has become an idea as-if-shared, as attested to by the repeated use criterion.

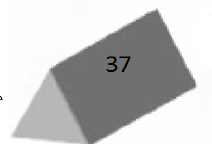
*Episode 11: Rectangular Prism*

|  |  |  |
| --- | --- | --- |
| **Line Number** | **Speaker** | **Utterance** |
| 102 | Facilitator | Wonderful! Next object. Is Object Number 37 a pyramid? |
| 103 | Teacher 11 | No, the faces are not formed exclusively of triangles. Therefore, the object does not meet all the parameters of the pyramid. |

In response to the facilitator’s question about the object on display (102), Teacher 11 raises Argument 16, which claims that the object is not a pyramid. She provides a warrant and backing (103).

Argument 16: (102-103) The object is a prism.

Claim 16: Not a pyramid (103)

Data 16: The object on display 

Warrant 16: The sides are not comprised exclusively of triangles. (103)

Backing 16: The object does not meet all the parameters of the pyramid. (103)

The use of the phrase ***“the pyramid does not meet the parameters”*** (103) is repeated during the course of this session’s discussion as backing in Argument 16, after appearing as Backing 11a in Argument 11 (81). This is further proof that the phrase has become an idea as-if-shared, as attested to by the repeated use criterion.

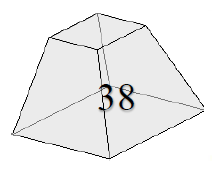
*Episode 12: Truncated Pyramid*

|  |  |  |
| --- | --- | --- |
| Line Number | Speaker | Utterance |
| 104 | Facilitator | Correct. The next object. Is Object Number 38 a pyramid? |
| 105 | Teacher 1 | No. Because the apex is not shared, and it does not meet the parameters of the pyramid. |

In response to the facilitator’s question about the object on display (104), Teacher 1 raises Argument 17 which claims that the object is not a pyramid, and provides a warrant (105).

Argument 17: (103-105) This is a truncated pyramid.

Claim 17: Not a pyramid (103)

Data 17: The object on display .

Warrant 17: Because the apex is not shared, and it does not meet the parameters of the pyramid. (105)

The use of the phrase ***“the pyramid does not meet the parameters”*** (105) repeats for the third time during the course of this session’s discussion as a warrant in Argument 17, after having appeared previously as backing in Argument 11 (81). This is further proof that the phrase has become an idea as-if-shared, as attested to by the shift of position criterion. The use of the phrase ***“the pyramid does not meet the parameters”*** (109) is repeated once more during the course of this discussion as backing for Argument 19, after it appeared previously as backing in Argument 11 (81), as a warrant in Argument 16 (103), and as a warrant in Argument 17 (105). This is further proof that the phrase has become an idea as-if-shared, as attested to by the shift of position and repeated reuse criteria.

In conclusion, at this stage in the PD program’s ninth session, the kindergarten teachers and the facilitator reached conclusions about the critical attributes that pyramids must possess: an apex, an surface comprised of three triangular faces, and a polygonal base. The teachers arrived at mathematical ideas that function as-if-shared, as Rasmussen and Stephan (2002) described.

Our analysis of the argumentation dialog among the kindergarten teachers in whole-class discussions during the PD program’s ninth session revealed 19 arguments, some basic and some broader. The arguments were the basis for both correct and incorrect mathematical ideas, some of which functioned as-if-shared and some of which did not. Regarding the pyramid, seven correct mathematical ideas were raised in this session, all of which functioned as-if-shared. These ideas developed the practical mathematics that were identified in this session regarding the pyramid’s critical attributes. The study identified normative ways of reasoning for providing warrants and collected these for providing warrants in mathematical practices. Table 7.1.2 details the mathematical practices of the pyramid, which includes seven mathematical ideas that functioned as-if-shared in the whole-class discussion:

Table No. 2: Mathematical Ideas Comprising the Mathematical Practices – The Pyramid’s Critical Attributes (MP1)

|  |  |  |
| --- | --- | --- |
| The Practice | Concept Code | The Mathematical Idea that Functions As-If-Shared |
| The Pyramid’s Properties | NWR 1.1 | Components of the pyramid: triangular sides, a base and a top (apex) created by the convergence of these triangles. |
|  | NWR 1.2 | The pyramid’s base can be formed of any type of polygon. |
|  | NWR 1.3 | Defining the pyramid: Composed of an apex, a polygonal base, and triangular sides |
|  | NWR 1.4 | Any of a triangular pyramid’s triangles can also be the polygonal base of the pyramid. |
|  | NWR 1.5 | If the sides are not triangular then the object is not a pyramid. |
|  | NWR 1.6 | If the definition is not fulfilled for a particular object then the object is not a pyramid. |
|  | NWR 1.7 | If the definition is met for a certain object then the object is a pyramid. |

As shown in Table 7.1.2, the methods of warranting that function as-if-shared are numbered in the order they appeared in the teachers’ warrants in the ninth session. In other words, the way the ideas are displayed in the table mirror how they were presented in the whole-class discussion during the session. Thus, by employing DCA methodology to highlight the arguments that were raised during the whole-class discussion, we show that the teachers in this PD program interpreted the critical properties of pyramids in a normative fashion.

**4. Discussion**

The main goal of this study was to identify and understand the learning processes in a PD program for kindergarten teachers by tracking the development of the teachers’ mathematical ideas. We studied a mathematics session where the teachers worked as a whole-class community, on purposefully-designed sequences of tasks intended to facilitate the emergence of geometrical thinking through discussion.

To empirically evaluate and analyze the whole-class discussions, we used DCA methodology. The findings that emerged from this methodological analysis allowed us to study learning processes in the PD program by observing the development of mathematical ideas within a particular session. The theoretical framework and associated DCA methodology describe different but closely-related aspects of the classroom learning process. In this case, two separate components of classroom activity had to be considered together and coordinated, and, for this purpose, we combined the two theoretical frameworks and their methodologies.

There are advantages to using DCA methodology to examine how ideas function at the collective classroom level, and that enable the growth of ideas in the classroom community to be traced. The DCA methodology adopts Freudenthal’s (1991) position, defining mathematics as a human activity situated in the classroom, inseparable from the cognitive constructions made by students.

Other advantages include the fact that the DCA operates more on a macroscopic level and it frames more in activity terms. This also played out in the previous analyses where DCA methodology was used sequentially throughout the lesson to analyze diverse social settings. We tried to demonstrate this, using the data and analyses above.

**5. Theoretical contribution**

This analysis of learning processes during a mathematics PD program for kindergarten teachers on 3D shapes and objects sheds light on how the knowledge of the participants grew through argumentation discussions during whole class discussions. During the program, the participating teachers engaged in whole-class discussions to identify correct and incorrect examples of 3D shapes and provided warrants for their choices. These discussions led to a growth of mathematical ideas that functioned as-if-shared in a normative way, and related to the use of the critical properties of the shapes and concepts under discussion (Hershkowitz, 1990). The attention paid to the critical properties of a concept can help with planning diagnostic tasks, and when constructing tasks to advance knowledge about the concept. Therefore, we recommend that PD programs for kindergarten teachers emphasize the importance of choosing correct and incorrect examples of a certain class of objects, from a selection of objects that teachers will present to pupils while structuring their knowledge about the concept in question. This study of kindergarten teachers demonstrates how a fairly brief intervention of only thirty hours, planned according to specific principles, led to the development of seven correct mathematical ideas, all of which functioned as-if-shared. In this case, the ideas developed mathematical practices relating to the critical properties of the pyramid.

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