**Students' Conceptions of Congruent and Similar Triangles Definitions**

**Abstract**

There is relativity little research on knowledge about teaching and learning of theorems and definitions of congruent and similar triangles. The study reported here addresses high- school students' conceptions of mathematical definitions of congruent and similar triangles. The study involved 120 students. The findings indicate that many of the participants differentiated between definitions and theorems. Many of the participants did not always accept the congruent- and similar-triangles theorems as formal definitions of congruency and similarity. From the participants’ explanations of their responses and from the interviews performed, it appears that two issues prevented some participants from accepting or preferring these theorems as definitions: a concern for *uniformity*, specifically that there is only one known and accepted definition of each concept, and a focus on the *essence* of the concepts, specifically that the essence of the concepts of similarity and congruency lies primarily in the lengths of the sides of a triangles. The students who accepted these theorems as formal definitions explain it by the equivalence and that the theorems include necessary and sufficient attributes. The study revealed a correlation between the participants’ responses about accepting theorems as formal definitions and their reasons raised from their explanations.

*Keywords*: congruent-triangles theorems, formal definition, similar-triangles theorems.

**1. Introduction**

Many studies have investigated how people understand the process of defining concepts and the need for definitions (e.g., Choi and Kim 2013; de Villiers 2004; Zandieh and Rasmussen 2010). Other studies have investigated how students understand the definitions of some geometric figures, such as triangles and quadrilaterals (e.g., Fujita and Jones 2007; Kaur 2015; Usiskin et al. 2008). Some studies have investigated students’ perceptions regarding congruent and similar triangles (e.g., Gonzalez and Herbst 2009). However, in the research literature, we do not find any work that has clearly focused on students' conception of congruent and similar triangles definitions. The current study attempts to fill in this gap in the professional literature. This study examined students' conceptions regarding the definitions of similar and congruent triangles. The results of this research add to our knowledge regarding these concepts and, in particular, the characteristics of these definitions among students.

**1.1. Mathematical definitions**

Many researchers have argued that definitions play a central role in mathematical theorems and proofs (Author et al. 2014; Moore 1994; Pimm 1993; Smith 2010; Weber 2002). Studies have claimed that definitions play a central role in understanding the construction of the meaning and the essence of mathematical concepts (e.g., Okazaki 2013; Wilson 1990). Okazaki (2013) reported on five ways in which fifth-graders enhanced their familiarity with mathematical definitions: (1) by understanding the meaning of the identification of geometric figures, (2) by constructing examples from non-examples and making comparisons to justify those constructions, (3) by recognizing equivalent combinations, (4) by using counterexamples to evaluate undetermined cases and (5) by conceiving of figures as relations beyond the given actualities. Borasi (1992) claimed that definitions serve as a tool for creating uniformity in the meaning of mathematical concepts and for communication among people.

According to van Hiele and van Hiele’s (1958) theory about the development of geometric thinking, at the informal deduction level (third level), the learner understands the importance of precise definitions and how a particular attribute derives from another. Tall and Vinner (1981) regarded definitions as the words used to specify a particular concept. Later, in Vinner's (1991) paper on the role of definitions, five assumptions were made: concepts are acquired by their definition, students use definitions to solve problems and to prove theorems, definitions must be minimal, definitions must be elegant and definitions are arbitrary. Zaslavsky and Shir (2005) distinguished between the roles and features of mathematical definitions; they classified the features of mathematical definitions as imperative features and optional features. They listed the imperative characteristics that must exist in mathematical definitions: There is no inherent contradiction between the concept attributes; there is no ambiguity; there are no changes under one or another representation of the concept; definitions are formulated in a hierarchical (based on previous concepts) and noncircular manner. They mentioned that the most notable example of a controversial optional feature is the requirement that a mathematical definition be minimal. A definition is considered to be minimal if it is economical, with no superfluous, unnecessary conditions or information. In addition to the imperative criteria, Van Dormolen and Zaslavsky (2003) demanded further criteria for logical necessity, specifically: a criterion of existence, that there exists an instance of such concept; a criterion of equivalence, that when one gives more than one formulation for the same concept definition, he or she must prove that all of those formulations are equivalent; and a criterion of axiomatization, which implies that a definition fits into and is part of a deductive system. They also noted two optional criteria for definition: a criterion of degeneration and a criterion of elegance, which states that in a case of two equivalent definitions, the one that looks nicer, needs fewer words or symbols, or uses more general basic concepts should be preferred.

De Villiers (2004) distinguished between two different kinds of defining: *descriptive defining* and *constructive defining*. In descriptive defining, the image of the concept is developed before a definition is formulated for it, based on an appropriate subset of the total properties from which all of its properties can be deduced. That subset serves as a definition and the remaining properties logically derived from it are theorems. In constructive defining, a known definition of a concept is changed through exclusion, generalization, specialization, replacement or addition, to construct a new definition of the concept. In this case, the definition of the new concept precedes the further exploration of the concept and the development of the image of the concept.

One of the characteristics of the definitions that mathematicians and mathematical educators use is that a certain definition of a concept is equivalent to other definitions of the same concept (Harel et al. 2006; Usiskin et al. 2008). These definitions are arbitrary because they are man-made (Vinner 1991). A particular concept is defined by selecting one statement of a set of logically equivalent statements; this means that each of the statements in that set could be used as a legitimate definition for the particular concept.

Many studies have shown that students can have trouble understanding the structure of definitions and their meanings (de Villiers et al. 2009; Foster 2014; Fujita and Jones 2007; Hershkowitz 1987; Marchis 2012; Pickreign 2007).

Linchevsky et al. (1992), de Villiers (1998), de Villiers et al. (2009) and Foster (2014) all reported on the tendency of students and pre-service teachers to make a long list of all of the attributes of a concept. These long descriptive definitions are indeed correct, but many Math educators prefer mathematical definitions to be minimal and elegant, as indicated above (e.g., Leikin and Winicky-Landman 2001; Van Dormolen and Zaslavsky 2003; Vinner 1991). On the other hand, there are those who, in certain cases, prefer non-minimal definitions (de Villiers 1998; Pimm 1993; Van Dormolen and Zaslavsky 2003; Zaslavsky and Shir 2005).

Leikin and Winicky-Landman (2001) investigated Math teachers (not in the context of Geometry) and found that many high school Math teachers do not notice that a particular concept can be defined by a number of equivalent definitions. Vinner (1991) referred to the defining process within mathematical deductive theory as follows:

*Typically, one starts with well-known notions and well-known theorems and proceeds by defining new notion and by proving new theorems.* (p. 65(

Vinner (1991) added that teachers must take into account the concept acquisition and the logical reasoning that are part of this process. Harel et al. (2006) related to the difficulty of assessing the accuracy and efficiency of formal proofs and the difficulty of mathematical definitions, and stated the following:

*As it is commonly difficult for students to appreciate the precision and economy of thought afforded by formal proof, it is likely that they experience similar difficulty with mathematical definitions.* (Harel et al. 2006, p. 153)

Van Dormolen and Zaslavsky (2003) argued that when a person gives more than one definition for a single concept, that person needs to choose one of those formulations as the definition and consider the other formulations as theorems that must be proven to be equivalent definitions of the same concept. Türnüklü et al. (2013) found that personal definitions of mathematical concepts are often based on the names given to those concepts, which can lead to many misjudgements. In addition, Author et al. (2014) reported that the name *parallelogram* affects students’ proving processes. It seems that the influence of the name on the participating students’ conceptions of definitions hindered those students from considering some of the equivalent and alternative definitions of a parallelogram as accurate definitions.

**1.2. Definitions and theorems**

Fishbein (1994) referred to definitions and theorems as components of mathematics as a formal science. In his words:

*This (the formal aspect) refers to axioms, definitions, theorems and proofs. The fact is that all these represent the core of mathematics as a formal science.* (p. 231).

Freudenthal (1968) mentioned that turning definitions into theorems and theorems into definitions are some of the most fruitful activities for mathematician and the students expected to enjoy these fruits. Also van Dormolen & Zaslavsky (2003) claimed that theorems could be a definition, when we have two equivalent definitions, one has to chose one as a definition and formulate the other as a theorem.

Van Dormolen and Zaslavsky (2003) enumerated the features needed for a mathematical definition to be a good definition. One of those requirements is that the definition correspond to the deductive system to which it belongs and that it be a fundamental part of that system. These deductive systems include axioms, theorems and proofs.

Vinner (1991) mentioned that, in the classroom, Math teachers might develop a sequence of definitions, theorems and proofs as a skeleton for Math courses. In the case of congruent and similar triangles, the same conceptual skeleton can be used. After proving the congruent- and similar-triangles theorems and highlighting the necessary and sufficient conditions of those theorems, teachers can use these theorems to solve problems. They can use them to identify, classify, and prove congruent and similar triangles. That is, these theorems fulfil the role of definitions with regard to these concepts (Moore, 1994; Vinner, 1991; Weber, 2002).

Studies reported that students interpret the content of theorems incorrectly; Hazzan and Leron (1996) reported that students were "naive" and used theorems as a vague "slogans," that is, as a way of answering test questions while avoiding the need for understanding or excessive mental effort. In addition, Selden and Selden (2008) reported that undergraduate students often interpret the content of theorems incorrectly and have difficulty unpacking the logical structure of informally stated theorems.

**1.3. Congruent and similar triangles**

The definition of congruent triangles is "Two triangles, △ABC and △A’B’C’, are congruent if and only if their corresponding angles are the same size and the lengths of their corresponding sides are equal." And the definition of similar triangles is "Two triangles, △ABC and △A’B’C’, are similar if and only if their corresponding angles are the same size and the lengths of their corresponding sides are proportional." These definitions are non-minimal definitions therefore we can mention less attributes and deduce the rest of the attributes. There are theorems in which we mention minimal attributes in order to reach congruence-triangles or similar-triangles. These theorems focus on the sets of necessary and sufficient attributes that ensure congruency or similarity of triangles. For example, "Two triangles, △ABC and △A′B′C′, are congruent if and only if two angles and the inscribed side are equal."

The concept of congruent triangles is an important part of the basic knowledge needed to teach plane geometry (Luo and Lin 2007). The congruent triangles has a significant position because it links to similarity and the three conditions for triangle congruency are also the basis for proving other propositions (Jones et al. 2013). Wu (2005) claimed that the cases of congruency and similarity emphasize the need for definitions; without a mathematical definition of congruence and without a precise definition of similarity, learners cannot properly understand other topics in geometry, such as length and area.

Gonzalez and Herbst (2009) proposed the following four conceptions about congruency: the perceptual conception of congruency, the measure-preserving conception of congruency, the correspondence conception of congruency and the transformation conception of congruency. We can use these same conceptions and adjust them for similarity.

Jones and Fujita (2013) reported that many Grade 8 students in Japan have not fully developed their correspondence conception of congruency. They added that about 40% of those students are not sure how to use congruent triangles to deduce conclusions.

Many studies have investigated students’ perceptions of the congruent-triangles theorems (Hadas et al. 2000; Hoyles 1998; Jones et al. 2013). These studies examined students’ understandings that the conditions in the congruent- and similar-triangles theorems are actually necessary and sufficient conditions for producing and constructing congruent or similar triangles. In the congruent- and similar-triangles theorems, we use necessary and sufficient attributes to deduce other attributes. These theorems can serve as formal definitions for congruent and similar triangles, as they include all of the imperative and optional features of mathematical definitions, such as parsimony and elegance (Van Dormolen and Zaslavsky 2003; Zaslavsky and Shir 2005). For example, it is possible to define two similar triangles as two triangles that are similar if and only if two angles of one triangle are congruent to the corresponding two angles of the other triangle. Based on those attributes, we can deduce the four remaining attributes, which also exist in similar triangles. An understanding of these theorems strengthens the understanding of Math as deductive theory and also strengthens the understanding of the logical necessities of mathematical context (Okazaki 2013; Van Dormolen and Zaslavsky 2003; Vinner 1991). However, from a pedagogical perspective, adherence to only this minimal definition may impair students’ understanding of the concept of similar triangles (Zaslavsky and Shir 2005).

In the research literature, we have not found any studies that have clearly focused on the definitions of congruent and similar triangles among students. If the theorems related to these concepts can function as definitions, why is this so? Revealing these reasons could shed light on the conception congruent and similar triangles'' definitions, as perceived by students.

**1.3. Research rationale and goals**

When students learn that the attributes included in these theorems are sufficient to construct two similar or congruent triangles. Some students learn the proofs of these theorems, indicating that they are aware that these theorems contain necessary and sufficient attributes. Furthermore, the existence of more than one theorem might emphasize the equivalence between these theorems. These activities can sharpen the logical structure of the mathematical definition, the elegance of these definitions (when fewer words and symbols are used) and the minimalism of these definitions (when minimal attributes are used in the proofs).

However, in the literature, I could not find any studies that have clearly focused on the definitions of congruent and similar triangles, the relationship between congruent-triangles theorems and similar-triangles theorems, or how these concepts are defined by students. One of the abilities expected at the fourth level of van Hiele and van Hiele’s (1958) hierarchy is the ability to understand definitions, axioms, theorems and proofs as connected units in a deductive structure. Therefore, the explicit goal of this study is to investigate the students' conception of congruent and similar triangles definitions. the reasons behind the acceptance or non-acceptance of theorems of congruent- and similar-triangles as definitions of those concepts could give us an insight about the characteristics of mathematical definitions as perceived by students.

I chose to deal with tasks related to congruent and similar triangles for a number of reasons. First of all, these concepts are very familiar to the participants; they learn them at the junior-high level, toward the beginning of their lessons about proofs and deduction. Second, the logical structure that exists between the attributes of these concepts makes them easy subjects with which to conduct this type of study. Finally, there are very famous and useful theorems related to these concepts. Since students are expected to use those theorems in identification, construction and proving tasks, I want to know whether these tasks affected the participants' conceptions about congruent and similar triangles definitions.

**2. Method**

The study reported here addresses high- school students conception regarding the definition of similar triangles and the definition of congruent triangles. It focuses on how these definitions and their logical structures (i.e., that each definition contains necessary and sufficient attributes) are understood. This study aimed to answer the following questions: How the participants define the similar-triangles and congruence-triangles concepts? What are the characteristics of the definitions of congruent and similar triangles, according to the participants?

**2.1. Participants**

The research population was students from a regional Arab high school in central Israel. The sample included 120 out of 340 10th-grades students in the school, who studied geometry with four different teachers in four parallel 4-points groups. (In Israel, the levels of mathematics in matriculation exams are 3, 4, and 5 points, the latter considered the highest achievement). Two of the teachers had a first degree in mathematics and two of had second degree in mathematics education. All had more than fifteen years’ experience in teaching mathematics. The participants studied units about all the congruent triangles theorems and about the first similar triangles theorem (angle, angle) in 9th grade and in 10th grade they studied the other two similar triangles theorems (angle, side, angle; side, side, side).

**2.2. Instruments and procedure**

The research instruments included a two stages questionnaire developed especially for this study and semi-structured interviews (the questionnaire presented as Appendix 1). In the first stage the students asked to define the similar-triangles and congruent-triangles concepts, the second stage questionnaire examined the participants' perspectives of the mathematical definitions of congruent and similar triangles. The first stage questionnaire was constructed in order to skim the knowledge accumulated of students’ definition about similar-tringles and congruence- triangles concepts. The construction second stages questionnaire and the interviews followed it were based on the analysis of students' responses from the first stages. This analysis gave us the opportunity to look at aspects which I didn’t pay attention about. For example, we didn’t know that the majority of the students who gave minimal definitions of the similar-triangles and congruence-triangles concepts, their definitions were based on sides only, therefore in the second stage questionnaire I included one minimal definition which based only on sides and the other minimal definition based only on angels. The second stage included two tasks: one concerning congruent triangles and one concerning similar triangles (see Fig 1.)

|  |
| --- |
| 1. Two students debated how similar triangles should be defined. Sami said, "Two triangles, △*ABC* and △*A’B’C’*, are similar if and only if their corresponding angles are the same size and the lengths of their [corresponding sides](https://en.wikipedia.org/wiki/Corresponding_sides) are [proportional](https://en.wikipedia.org/wiki/Proportionality_(mathematics))." Rami argued that Sami's definition included a superfluous condition and suggested the following definition: "Two triangles, △*ABC* and △*A′B′C′*, are similar if and only if they have two congruent angles."   Which definition/s is/are correct? Explain your answer!   1. Two students debated how to define congruent triangles. Sami said, *"**Two triangles, △ABC and △A’B’C’, are congruent if and only if their corresponding angles are the same size and the lengths of their* [*corresponding sides*](https://en.wikipedia.org/wiki/Corresponding_sides) *are equal."* Rami said that there was a superfluous condition in Sami's definition and suggested the following definition: *"Two triangles, △ABC and △A′B′C′, are congruent if and only if all three of their side are equal."* Which definition/s is/are correct? Explain your answer! |

Fig. 1 The second stage questionnaire

In each task, two definitions were given: Rami's definition and Sami's definition. All of the definitions were correct, but Sami's definitions were non-minimal while Rami’s were minimal. Sami defined similar triangles as follows: *"Two triangles, △ABC and △A’B’C’, are similar if and only if their corresponding angles are the same size and the lengths of their corresponding sides are proportional."*

He defined congruent triangles as follows: "*Two triangles, △ABC and △A’B’C’, are congruent if and only if their corresponding angles are the same size and the lengths of their corresponding sides are equal."*

Rami’s minimal definitions included necessary and sufficient attributes to lead to all of the critical attributes to which Sami referred. Furthermore, Rami's definitions were based on congruent-triangles or similar-triangles theorems. Rami defined similar triangles as follows: *"Two triangles, △ABC and △A′B′C′, are similar if and only if they have two congruent angles."*

He defined congruent triangles as follows: *"Two triangles, △ABC and △A′B′C′, are congruent if and only if all three of their sides are equal."*

I chose these tasks, one based in theorem and which includes only sides and one that includes only angles, in order to investigate whether this difference would be a factor in the participants’ responses. Specifically, I wanted to investigate whether the participants would accept a definition that included only sides and reject a definition that related only to angles. In these tasks, the participants were asked to reflect on the proposed answers. This gave them the opportunity to use critical thinking. In addition, the participants were asked to explain their responses and those explanations revealed some of their views and knowledge regarding definitions and theorems. These explanations and the interviews that followed the completion of the questionnaire helped me to address the goal of the research, that is, the participants’ conception of the definitions of congruent and similar triangles.

These tasks are only representative tasks; other tasks could be designed based on other congruent-triangles and similar-triangles theorems. For example, if we wanted another task related to the concept of similar triangles, we could ask whether the statement, *"Three sides of one triangle are proportional to three sides of other triangle"* is a definition of similar triangles.

We could also ask whether the statement, *"Two congruent angles and the included sides of one triangle are equal to the corresponding parts of the other"* is a definition of congruent triangles.

The first stage questionnaire was distributed during one mathematics lesson, two weeks after collecting the first stage questionnaire and analysing it the second stage questionnaire was distributed. All but three students volunteered to fill out the questionnaires, these three students have very low mathematical achievements, they asked their teachers not to participate in the study and therefore their teachers have allowed them not to answer the questionnaire. The participants completed to answer the first stage questionnaire within 10 minutes and the second stage questionnaire within 15 minutes.

After administering the questionnaires and analyzing the responses, I interviewed eleven participants who provided answers and explanations that were representative of the difficulties reported by the majority of participants. For example, some of the interviewed participants did not accept the two theorems of similar and congruent triangles as formal definitions of those concepts. Others didn’t accept the similar-triangles theorem which include only angels as formal definitions, but accepted the congruent-triangles theorem that included three equal sides as formal definition. Each interview lasted about 17 minutes. The structured part of the interview included the same questions that were asked in the questionnaire and the unstructured part included questions formulated according to the interviewees’ responses. The goal of the interview was to examine whether the participants were indeed certain of their answers and to clarify points that were not addressed by the questionnaire or which required deeper examination. For example, in the questionnaire, I wanted to examine whether the participants would accept a minimal definition of similar triangles that included only angles as a formal definition, and I did not check whether the participants accepted other minimal definitions of the same concept which include only sides. In the interview, I had the opportunity to do so, thereby adding important nuance to the questionnaire findings. I chose this method because I wanted to address trends and tendencies that might arise from the questionnaire results in a setting in which I would be able to directly address participants in a more focused manner (two representive interviews are presented as Appendix 2).

**2.3. Data analyses**

The students’ responses were analysed using both qualitative and quantitative methods. For analysing the explanation about the student's responses, I used a qualitative coding method (Salanda, 2015) that is close to grounded theory (Glaser & Strauss, 1967). We used deductive codes derived from a theoretical perspective (Charmaz et al., 2007) and inductive codes for the themes not present in existing research about geometric education. Using the deductive codes, I characterized the answer of each participant according to its satisfaction of the aspects of definition. I stopped constructing categories when there were no new categories in the students' responses. All of the codes from the questionnaires were entered into the SPSS program and frequencies were calculated. Next, a Pearson chi-squared test was performed to see whether there were any statistical significance for the relation between the students' explanations and their answers to accepting definitions.

3. Results

In this section, I describe participants’ answers in detail, based on an analysis of the three tasks on stage 1 and stage 2.

**3.1. Task 1-stage 1: definition of similar-triangles and congruent-triangles concepts**

In Task 1, participants were asked to define the concepts congruent-tringles and similar-triangles concepts. Our analysis revealed five categories of responses to Task 1, as described below:

1. Examples of non-economical definitions:

* Two triangles are similar if and only if their corresponding angles are the same size and the lengths of their corresponding sides are in the same proportion.
* Two congruent triangles if their corresponding angles are the same size and the lengths of their corresponding sides are equal.

1. Examples of economical definition including only sides:

* Two triangles are similar if their sides are in the same proportion.
* Two triangles are congruent if all three of their side are equal.

1. Examples of economical definition including angels:

* Two triangles are congruent if two of their side are equal and the inscribed angle between them is equal.
* Two similar triangles if two of their side are proportional and the inscribed angle between them is equal.

1. Examples of non-sufficient definition:

* Two congruent triangles if their angles are equal.

1. Examples of intuitive definition:

* Two congruent triangles if each of the triangles cover the other.
* Two triangles are similar if we have the same shape but of different size.

Table 1

*Responses to Task 1*

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Non economical definition | Economical definition including only sides | Economical definitions including angels | Non-sufficient definition | | Intuitive definition | | No responses | Total |
| Similar triangle concept | 67  55.83% | 35  29.16% | 3  2.5% | 8  6.66% | 4  3.33% | | 3  2.5% | | 120  100% |
| Congruent triangle concept | 69  57.5% | 40  33.33% | 5  4.16% | - | 3  2.5% | | 3  2.5% | | 120  100% |

In Table 1, we can see clear tendency for giving non-economical definition for congruent and similar triangles concepts. 57.5 % gave non-economical definition for congruent triangles concept and about 56% gave non-economical for the similar triangles concept. I addition, when we talk about the student who gave economical definition and comparing with the students who gave economical definitions including angels we found that the vast majority of them gave economical definition which include only sides (about 33% who gave for congruent triangles concept and about 29% who gave for similar triangles concept). Moreover, we can see that very few students who gave intuitive definitions.

**3.2. Task 2-stage 2:** **Based on the similar-triangles theorem (angle, angle)**

In Task 2, participants were asked to choose between Sami's non-minimal definition that two triangles are similar if and only if their corresponding angles are the same size and the lengths of their corresponding sides are proportional and Rami's definition that two triangles are similar if and only if they have two congruent angles. Our analysis revealed six categories of explanations responses to Task 1 and Task 2, as described below:

1. Examples of difference between definition and theorem:

* Sami's argument is a definition and Rami's is a theorem and there is a deference between definition and theorem.
* Rami used a theorem and not a definition.

1. Examples of uniform definition:

* There is one accepted definition.
* Sami's definition is the accepted one for the concept of similarity of triangles, with the necessary attributes mentioned in detail.
* This the known definition for all the students and teachers.

1. Examples of mathematical essence of the concept:

* Sami gave a long definition that includes all of the conditions of congruency, but Rami's definition is also accepted as a formal definition; it emphasizes the meaning of the concept.
* Rami's definition is also accepted as a formal definition; it emphasizes the meaning of the concept.
* Sami describes the meaning of similarity and this is a good.

1. Examples of non-sufficient definition:

* Rami's include non-sufficient attributes. He didn’t mention the all attributes.
* Rami have to mention more attributes about similarity.

1. Examples of equivalent definition:

* From equal sides we can deduce equal angles, but it is more accurate to use Sami's definition
* From one we can deduce the other. These are equivalent.

1. Necessary and sufficient attributes:

* He uses a congruence theorem. The other describes the congruent and this is a good, but very long definition.
* Without a doubt, Rami is right; it is sufficient that two angles from one triangle be equal to two angles in another triangle to say they are similar triangles.

Table 2

*Responses to Task 2*

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Difference between definition and theorem | Uniform definition | mathematical essence of the concept | Non-sufficient definition | | Equivalent definition | | Necessary and sufficient attributes | Total |
| Only Sami's definition is correct. | 16  13.33% | 24  20% | 18  15% | 8  6.67% | - | | 2  1.67% | | 68  56.67% |
| Only Rami's definition is correct. | - | 1  0.83% | - | - | 3  2.5% | | 8  6.67% | | 12  10% |
| Both definitions are correct. | 1  0.83% | 3  2.5% | 2  1.67% | 1  0.83% | 17  14.17% | | 16  13.33% | | 40  33.33% |
| Total | 17  14.17% | 28  23.33% | 20  16.67% | 9  7.5% | 20  16.67% | | 26  21.67% | | 120  100% |

The Pearson chi-squared test revealed a correlation between the participants’ responses about the acceptance of the definitions and the explanations they gave for their responses(chi-square (10, N=120, p=.000<0.01)). As evident in Table 2 there was a tendency for accepting only Sami's non-economical definition for similar triangles concept. About 57% among all the participants claimed that only Sami's non-economical definition is right, about 35% among them said that there is a uniform definition. About 27% of the students who claimed that only Sami's non-economical definition is right said that it emphasized the mathematical essence of the concept. these participants argued that the definition must reveal the mathematical essence of the concept. And 24% among the students who accept only Sami's non-economical definition said that there is a difference between definition and theorem.

Only about 33% among all the participants claimed correctly the both definitions are right. about 83% among them said that the definitions are equivalent or that Rami's definition include necessary and sufficient attributes for defining similar triangles concept. These participants' explanations indicate that they behaved as expected for van Hiele and van Hiele’s (1958) third level. For example, Tamir explained, *"Sami's definition derives from Rami's definition,"* . Tamir understood the equivalence of the definitions and understood that the theorem of similar triangles (angle, angle theorem) provides a minimal definition for similar triangles.

Yossif’s responses were very interesting and so he was interviewed. For Yossif, the equality of angles does not fully reflect the meaning of the concept of similar triangles. In the interviews we have the opportunity to investigate whether replacing the definition based on the similar-triangles theorem (angle, angle) to other theorem which based on the other similar-triangles theorem (side, side, side) will cause to Yossif to change his response and accept the theorem which based only on sides as definition for similar-triangles concept.

Interview 1: Yossif

Interviewer: *Can we use the criterion "two angles of one triangle have the same measure as two angles of another triangle" to identify two similar triangle?*

Yossif: Yes, we can use it and we used it in order to do tasks in geometry.

I: *In the questionnaire, you claimed that Rami’s definition […] is wrong.*

Y: *Yes, Rami's is not right definition.*

I: *Although it describes similar triangles?*

Y: *Yes, because it does not give us the essence and the meaning of the concept.*

I: *Could the attribute "three sides are proportional in two triangles" be a classification criterion for similar triangles?*

Y: *Yes, this is the theorem. And we sort similar- triangles by it.*

I: *One student defined similar triangles as follows: "Two triangles are similar when all of their corresponding sides have lengths of the same ratio." Can you accept it as a correct definition?*

Y: *Yes, I can accept it as a correct definition, because in this definition, the essence of the concept is clear.*

I: *Does the [aforementioned] statement equivalent to the statement "two angles of one triangle are equal to two angles of the other triangle"?*

Y: *Yes, because from one theorem we can conclude the other theorem.*

I: *Why one theorem you accepted as definition and the other you didn’t accept?*

Y: *Because of the essence of the concept. One gave us the essence and the other not.*

Yossif does not understand that all theorems of similar triangles provide us with a minimal definition for similar triangles. It is important for him that the definition include the attributes that embody the essence of the concept (i.e., the sides are proportional). Yossif accepted *"three sides are proportional in two triangles"* and *"two angles of one triangle have the same measure as two angles of the other triangle"* as criteria for classifying similar triangles, but he accepted only the first criterion as a definition of similar triangles. He claimed that only criteria that highlight the essence of the concept can constitute a formal definition. Thus, Yossif failed to reach van Hiele and van Hiele’s (1958) fourth level, at which the learner understands the function of mathematical definitions, in terms of identifying and classifying examples and non-examples of a given concept. He also failed to understand the concept of necessary and sufficient attributes and the equivalence of formal mathematical definitions.

To conclude, knowledge of the similar-triangles theorem did not guarantee that a participant would accept it as a formal definition for similar triangles.

**3.3. Task 3-second stage: Based on the congruent-triangles theorem (side, side, side)**

In Task 3, participants were asked to choose between Sami's definition that two triangles are congruent if and only if their corresponding angles are the same size and the lengths of their corresponding sides are equal and Rami's definition that two triangles are congruent if and only if they have three equal sides. The participants’ responses to Task 3 are presented in Table 3.

Table 3

*Responses to Task* 3

|  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Difference between definition and theorem | Uniform definition | mathematical essence of the concept | Non-sufficient definition | Equivalent definition | | Necessary and sufficient attributes | No explanation | Total |
| Only Sami's definition is correct | 15  12.5% | 12  10% | - | 8  6.67% | - | - | | 1  0.83% | 36  30% |
| Only Rami's definition is correct | - | 10  8.33% | 7  5.83% | - | 2  1.67% | | - | 1  0.83 | 20  16.67% |
| Both definitions are correct. | - | 6  5% | 25  20.83% | - | 11  9.17% | | 15  12.5% | 3  2.5% | 60  50% |
| No response | 3  2.5% | 1  0.83% | - | - | - | | - | - | 4  3.3% |
| Total | 18  15% | 29  24.17% | 32  26.67% | 8  6.67% | 13  10.83% | | 15  12.5% | 5  4.17% | 120  100% |

The Pearson chi-squared test revealed a correlation between the participants’ responses about the acceptance of the definitions and the explanations they gave for their responses (chi-square (18, N=120, p=.000<0.01)). In Task 3, 30% (36) of the participants claimed in-correctly that only Sami's non-economical definitions is right. about 42% of them referred in their explanations to the claim that there is a deference between definitions and theorems. For example, Soli's explanation that *"Sami mention a definition and Rami mention a theorem, and there is a deference between them*". About 33% of them based their explanations to the uniformity of definition, like Sewar which claimed that *"Sami's definition is what accepted in all the text-books and the teachers as a definition"*. About 22% of the students who claimed in-correctly that only Sami's non-economical definitions is right, claim that Rami's definition includes non-sufficient attributes, these students didn’t understand the meaning of congruent-triangles theorem.

Half of the participants claimed correctly that both definitions are right and accepted the economical definition based on the congruent triangles' theorem (side, side, side) as valid definition for congruent triangles concept. About 42% of them claimed that Rami's definition emphasizes the essence of the concept. About 43% of the participants who claimed correctly that both definitions are right based their explanations on the argument that Rami's definition include necessary and sufficient attributes.

The surprising result was that about 17% of the participants claimed incorrectly that only Rami's economical definition based on the congruent triangles theorem (side, side, side) is right. half of them based their explanation on the uniformity of definition. 35% of them explain that because of the mathematical essence of the concept.

Samir’s responses were very interesting and so he was interviewed. In the congruent-triangles and in the similar-triangles tasks, he answered in-correctly only one definition is correct, because of there is a difference between definition and theorem. In the interview I have the opportunity to investigate what he means about his explanation.

Interview 2: Samir

Interviewer: *did you accept the statement "two triangles, △ABC and △A′B′C′, are congruent if all the three side are equal" as a definition for congruent triangles.*

Samir: *No, I can't accept.*

I: *Although it is a theorem for congruency?*

S: *Yes. Because there is a difference between definition and theorem.*

I: *what is the difference?*

S: *Sami's is the definition and there is only one known and accepted definition, the other is theorem which one have to prove.*

I: *What are the roles of the congruent triangles' theorems.*

S: *To identify congruent triangles from non-congruent triangles …and help us to prove that two triangles are congruent.*

I: *So it is a base to decide whether two triangles are congruent or not congruent?*

S: *Yes.*

I: *And couldn’t be definitions?*

S: *No, it couldn't be a defintion .*

I: *I want to tell you that for one concept could be more than one definition.* *the definition must contain necessary and sufficient attributes and some of the roles of definitions are to sort examples and non-examples of the concept and to be base for proofs.*

S: *…..*

I: *Could you change your answer about the congruent triangles' theorems?*

S: *What to change?*

I: *If they could be definitions?*

S: *I think yes, they can be defintions.*

I: *And what about similar triangles' theorems?*

S: *although it difficult for me to accept the first theorem (angle, angle), but these theorems could be definitions for similar triangles concept.*

I: *Why it is difficult for you to accept the first theorem as definition?*

S: *Because it didn’t gave the meaning of the concept about proportion.*

Samir didn’t accept the theorems of congruency as formal definitions for congruency, Samir preferred the definitions that are known and accepted. The uniformity of definitions was an important issue for Samir on one side. And in the other side he think that there is deference between definition and theorems. At the moment Samir understand and know the roles and the features of mathematical definitions, behaviours which accepted in forth level of van Hiele and van Hiele’s (1958), he changed his answer and accepted the congruence triangles theorems as for definition for the concept. But he did not prefer the similar-triangles theorem, which includes two equal angles, as a definition for similar triangles concept because it doesn't give us the meaning of the proportional sides. He behaved like Yossif and demand a definition which gives us the meaning and the essence of the concept. Furthermore, Samir preferred the definitions that are known and accepted. The uniformity of definitions was an important issue for Samir.

**4. Discussion and Conclusions**

The present research aimed to examine the conceptions students regarding the definitions of similar triangles and congruent triangles. Investigating the acceptance or non-acceptance of theorems of congruent- and similar-triangles as definitions of those concepts could give us an insight about the characteristics of mathematical definitions as perceived by students.

Many of the participants isolated the defining process within mathematical deductive theory (Vinner 1991) and did not recognize that theorems might be definitions and that congruent-triangles and similar-triangles theorems are formulations of the definitions of those concepts and that after we prove them and accept them as true, they become definitions that are equivalent to the non-parsimonious definitions in which all of the attributes are mentioned (Van Dormolen and Zaslavsky 2003). These students didn’t enjoy the fruits by turning of theorems into definitions (Freudenthal, 1968). This behaviour is an example of the tendency to interpret the content of theorems incorrectly and the inability to unpack the logical structure of the theorem (Hazan and Leron, 1996; Selden and Selden, 2008). We can see this in the finding that participants’ knowing of congruent-triangles and similar-triangles theorems did not guarantee that they would accept those theorems as formal definitions of these concepts, an ability associated with the fourth level of van Hiele and van Hiele’s (1958) hierarchy. Furthermore, accepting one theorem as a formal definition of the concept did not guarantee accepting the other theorem as a formal definition. This confirms the findings of other studies regarding the equivalence of definitions (Harel et al. 2006; Usiskin et al. 2008). In this work, only 33% of the participants accepted the similar-triangles theorem: "Two triangles, △ABC and △A′B′C′, are similar if and only if they have two congruent angles" as a formal definition of similarity (see Table 2). The situation regarding the congruent-triangles theorem was better; 50% of the participants accepted the statement "Two triangles, △ABC and △A′B′C′, are congruent and only if all three side are equal" as a formal definition for the congruency (see Table 3). The fact that some of the students did not accept these theorems as formal definitions points to difficulties in understanding the characteristics, roles and the features of mathematical definitions. All of the similar-triangles and congruent-triangles theorems specify these concepts (Tall and Vinner 1981) and although these theorems have the imperative features of mathematical definitions — there is no inherent contradiction between the concept attributes, there is no ambiguity, there are no changes under one or another representation of the concept and the definitions are hierarchical and noncircular (Zaslavsky and Shir 2005) — these participants did not accept these theorems as formal definitions. These findings are in line with those of previous studies, which reported the tendency of students to make long lists of all of the attributes of a particular concept (de Villiers et al. 2009; Foster 2014; Linchevsky et al. 1992).

In accordance with the national curriculum, in the classroom, the students use the congruent- and similar-triangles theorems to solve classification, identification, and proving tasks. These theorems fulfilled the role of concept definitions, but many of the students did not accept these theorems as definitions. The participants were "naive" and used these theorems without making the mental effort to consider whether they could play the role of definitions (Hazzan and Leron, 1996).

In a comprehensive look at the findings from Task 2 and Task 3 the Pearson chi-squared test revealed a correlation between the participants’ responses about the acceptance of the definitions and the explanations they gave for their responses (sig. = 0.000, *p* < 0.01). The students who didn’t accept the similar-triangles and congruent-triangles theorems as formal definitions gave the explanations that there is a difference between definition and theorems, because of the essence of the concept or because of the uniformity of definition. The students who accepted these theorems as formal definitions gave the explanations about equivalent definitions or that the theorems include necessary sufficient attributes to define the concept. This what expected at the formal deductive level of van Hiele and van Hiele’s (1958) hierarchy.

Zaslavsky and Shir (2005) and Van Dormolen and Zaslavsky (2003) distinguished between two kinds of features of definitions: imperative features and optional features. The current study expanded upon those models, by adding another kind of feature to these models, namely, a non-critical feature. This additional optional feature is the *essence* of the concept. From the results, we can see that the participants accepted (see Tables 2 and 3 and interview with Yossif) or preferred (like in the interview with Samir) a formal definition that emphasized the essence and the meaning of the name of the concept and, therefore, they accepted a definition that included a description of the essence of the concept (Wilson 1990; De Villiers 2004; Okazaki 2013). It could be that the equality or proportionality of the lengths of the triangles’ sides is seen as more essential to the concepts of congruency and similarity than angles are. This fact could explain why more participants accepted the minimal congruent-triangles definition based on the congruent-triangles theorem, which contains only sides, rather than the minimal definition of similar triangles based on the similar-triangles theorem, which contains only angles. This finding confirms other research about the effects of the name of a concept on mathematical judgments (Author et al. 2014; Türnüklü et al. 2013).

The participants in another study gave greater weight to size than they did to correspondence (Gonzalez and Herbst 2009). We can see evidence when the vast majority of the students who gave or accepted economical definition based their definitions on sides only (see Table 1, Table 2 and Table 3) and in the interviews with Yossif. Yossif did not accept the statement *"Two triangles, △ABC and △A′B′C′, are similar if and only if they have two congruent angles"* as a formal definition. But, when I replaced it with another similar-triangles theorem, *"Two triangles are similar when all of their corresponding sides have lengths in the same ratio,"* he accepted that theorem as a correct definition. For Yossif the similar-triangles theorems cannot be used equally as definitions. This confirms other studies, which have shown the misunderstanding of two of the characteristics of mathematical definitions, namely that definitions are arbitrary (Vinner 1991) and that a certain definition of a concept may be equivalent to other definitions of the same concept (Harel et al. 2006; Usiskin et al. 2008).

An additional non-critical optional feature is the feature of uniformity. From the results, we can see that the participants accepted formal definitions based on the uniqueness of the concept definitions. They want to believe that for every concept there is only one accepted definition within the mathematics-education community, while all other statements are attributes. The participants understood that the subsets of conditions mentioned in the congruent- and similar-triangles theorems provide necessary and sufficient attributes to deduce the rest of the attributes (Hadas et al. 2000; Hoyles 1998; Jones et al. 2013), but still accepted the one uniform, accepted, non-parsimonious definition. This result is congruent with those of other studies that have reported about the inability to identify, accept or find equivalent definitions (Author et al. 2014; Harel et al. 2006; Leikin and Winicky-Landman 2001).

To conclude, the students' difficulties in understanding the characteristics and roles of mathematical definitions of geometric concepts affect their understandings of mathematical and geometric definitions. We can see this evidence in the interview with Samir when he understood that the definition must contain necessary and sufficient attributes and some of the roles of definitions are to sort examples and non-examples of the concept and to be base for proofs, he changed his response and accepted the theorem to be a formal definition. One can argue about what a good definition is, but we have to agree that when attributes are necessary and sufficient for classifying a concept, they can constitute a formal definition. For many participants, the essence of the mathematical concept (Mariotti and Fischbein 1997) is more important than the essence of the mathematical definition (Leikin and Winicky-Landman 2001). From a pedagogical point of view, one should not adhere to minimal definitions in the cases of similar triangles and congruent triangles because the non-minimal definitions emphasize the essence of these concepts (Zaslavsky and Shir 2005), but students must understand that the minimal definitions are correct and valid definition; this approach emphasizes the fact that Mathematics is a logical science. These results emphasize the need to avoid focusing only on descriptive definitions and avoid neglecting constructive definitions (de Villiers 2004). This also highlights the importance of addressing other situations reported by Okazaki (2013), in order to enhance learners’ familiarity with definitions, namely, conceiving figures as relations beyond the given actualities and recognizing equivalent combinations.

**4.1. Limitations, future directions and practical implications**

Future studies should involve larger and more diverse research populations. Future studies should also include teachers or pre-service teachers, as well as populations from different sectors of society and different parts of the world. This would allow us to determine whether cultural differences might affect the findings. It would also be interesting to use a different methodology, such as classroom observations, to gather more qualitative information about the population under study. It would be very interesting to see what emerges within the classroom discourse during such lessons, in order to learn about the thinking processes of both teachers and students, and most importantly, the interaction between those processes.

In this study, I attempted to investigate whether the participants accepted the congruent-triangles and similar-triangles theorems as formal definitions of those concepts. However, in the questionnaire, I included only one theorem for congruency and one theorem for similarity. In the interviews with Yossif, when I replaced the similarity, he changed their responses. I would like to examine the behavior exhibited by Yossif in a larger population. To that end, in future studies, it would be helpful to use a questionnaire that includes more than one similar-triangles theorem and more than one congruent-triangles theorem.

The results of this work may help researchers to plan educational studies that target particular characteristics of students' perspectives of definitions. To conclude, my recommendation in two connected directions: one is that students should have the experience in the process of defining, and two for the training of Geometry teachers is that teachers be exposed to the specific difficulties brought to light by this study. This will raise their awareness of the processes that lead to these difficulties and sensitize them, to help them cope with these issues in the teaching process. Creating such a mind set and motivation will help math teachers to diagnose and think through students’ difficulties and to perform better as teachers and should also improve student achievement.

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Appendix 1

Questionnaires

|  |
| --- |
| First stage   1. Please define the concept congruence-triangles. 2. Please define the concept similar-triangles.   Second stage   1. Two students debated how similar triangles should be defined. Sami said, "Two triangles, △ABC and △A’B’C’, are similar if and only if their corresponding angles are the same size and the lengths of their [corresponding sides](https://en.wikipedia.org/wiki/Corresponding_sides) are [proportional](https://en.wikipedia.org/wiki/Proportionality_(mathematics))." Rami argued that Sami's definition included a superfluous condition and suggested the following definition: "Two triangles, △ABC and △A′B′C′, are similar if and only if they have two congruent angles."   Which definition/s is/are correct? Explain your answer!   1. Two students debated how to define congruent triangles. Sami said, "Two triangles, △ABC and △A’B’C’, are congruent if and only if their corresponding angles are the same size and the lengths of their [corresponding sides](https://en.wikipedia.org/wiki/Corresponding_sides) are equal." Rami said that there was a superfluous condition in Sami's definition and suggested the following definition: "Two triangles, △ABC and △A′B′C′, are congruent if and only if all three of their side are equal." Which definition/s is/are correct? Explain your answer! |

Appendix 2

Interviews

Interview 1: Yossif

Interviewer: *Hi, I want to ask you some question about the questionnaire you responded.*

Yossif: *O.K*

I:*How can you define similar triangles?*

Y: *Two triangles have the same angels and the sides are proportional.*

I: *This can used in order to prove two similar triangles?*

Y: *Yes.*

Interviewer: *Can we use the criterion "two angles of one triangle have the same measure as two angles of another triangle" to identify two similar triangle?*

Y: Yes, we can use it and we used it in order to do tasks in geometry.

I: *In the questionnaire, you claimed that Rami’s definition […] is wrong.*

Y: *Yes, Rami's is not right definition.*

I: *Although it describes similar triangles?*

Y: *Yes, because it does not give us the essence and the meaning of the concept.*

I: *Could the attribute "three sides are proportional in two triangles" be a classification criterion for similar triangles?*

Y: *Yes, this is the theorem. And we sort similar- triangles by it.*

I: *One student defined similar triangles as follows: "Two triangles are similar when all of their corresponding sides have lengths of the same ratio." Can you accept it as a correct definition?*

Y: *Yes, I can accept it as a correct definition, because in this definition, the essence of the concept is clear.*

I: *Does the [aforementioned] statement equivalent to the statement "two angles of one triangle are equal to two angles of the other triangle"?*

Y: *Yes, because from one theorem we can conclude the other theorem.*

I: *Why one theorem you accepted as definition and the other you didn’t accept?*

Y: *Because of the essence of the concept. One gave us the essence and the other not.*

I:*lets go to the congruent triangles concept, "two triangles, △ABC and △A′B′C′, are congruent if all the three side are equal" are you accepted it as a definition for congruent triangles.*

Y: *Yes, I can accept it.*

I: *What about "two triangles, △ABC and △A′B′C′, are congruent if two angeles and the inscribed sides are equal" as formal definition?*

Y: *No, I can't accept it as defintion.*

I: *Why?*

Y: *Another time it didn’t give us the essence of the concept.*

I: *Although it is a theorem for congruency?*

Y: *No, it can't be a formal definition.*

I: *but it is written like definition?*

Y: *Definition have to give us insight about the concept.*

I: *What are the roles of the congruent triangles' theorems.*

Y: *To prove that the triangles are congruent triangles.*

I:*So it can classify congruent triangles?*

Y: *Yes.*

I: *And couldn’t be definitions?*

Y: *No.*

I:*Why?*

Y: *There is only two definitions, one includes all the attributes and the other gives the equal sides of the triangles and twice give the meaning of concept.*

I: *I want to tell you that the definition must contain necessary and sufficient attributes to sort examples and non-examples.* *Could you change your answer about the congruent triangles' theorems?*

Y: *What to change?*

I: *If they could be definitions?*

Y:*. No, only side, side, side could be a definition.*

I: *O.K thank you for your answers.*

Y: *Your welcome*.

Interview 2: Samir

Interviewer: *Hi, I just ask you to define similar triangles.*

Samir: *O.K, similar triangles are couple of triangles which have equal angles and the sides have same thing…no the sides are proportional.*

I: *This can used in criterion to sort similar triangles?*

S: *Yes, we can.*

I: *In the questionnaire, you claimed that only Sami’s definition […] is right.*

S: *Yes.*

I: *Can we use Sami's to prove similar triangles?*

S: *Yes, because it give only the similar-triangles.*

I: *So, it could be a classification criterion for similar triangles?*

S: *Yes.*

I: *why?*

S: *It’s the known theorem.*

I: *why it couldn’t be a definition for similar-triangles theorem?*

S: *Because there is one definition, Rami gave a definition and Sami gave a theorem, and there is deference between them.*

I: *What is the deference between them.*

S: *In the text-books Rami's is accepted as definition and Sami's as a theorem.*

I: *Do you think that for one concept there is only one definition?*

S: *Yes, I do.*

Interviewer: *did you accept the statement "two triangles, △ABC and △A′B′C′, are congruent if all the three side are equal" as a definition for congruent triangles.*

Samir: *No, I can't accept.*

I: *Although it is a theorem for congruency?*

S: *Yes. Because there is a difference between definition and theorem.*

I: *what is the difference?*

S: *Sami's is the definition and there is only one known and accepted definition, the other is theorem which one have to prove.*

I: *What are the roles of the congruent triangles' theorems.*

S: *To identify congruent triangles from non-congruent triangles …and help us to prove that two triangles are congruent.*

I: *So it is a base to decide whether two triangles are congruent or not congruent?*

S: *Yes.*

I: *And couldn’t be definitions?*

S: *No, it couldn't be a defintion .*

I: *I want to tell you that for one concept could be more than one definition. the definition must contain necessary and sufficient attributes and some of the roles of definitions are to sort examples and non-examples of the concept and to be base for proofs.*

S: *…..*

I: *Could you change your answer about the congruent triangles' theorems?*

S: *What to change?*

I: *If they could be definitions?*

S: *I think yes, they can be defintions.*

I: *And what about similar triangles' theorems?*

S: *although it difficult for me to accept the first theorem (angle, angle), but these theorems could be definitions for similar triangles concept.*

I: *Why it is difficult for you to accept the first theorem as definition?*

S: *Because it didn’t gave the meaning of the concept about proportion.*

I*: Thank you Samir for your answers.*

S*: Your welcome.*