ρ-EINSTEIN SOLITONS ON WARPED PRODUCT MANIFOLDS AND APPLICATIONS

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Abstract. The purpose of this research is to investigate how a ρ-Einstein soliton structure on a warped product manifold affects its base and fiber factor manifolds. First, many interesting properties of ρ-Einstein solitons are given. Then, some necessary and sufficient conditions on a ρ-Einstein soliton warped product manifold to make its factors ρ-Einstein soliton are examined. On a ρ-Einstein gradient soliton warped product manifold, necessary and sufficient conditions for making its factors ρ-Einstein gradient soliton are presented. Also, ρ-Einstein solitons on warped product manifolds admitting a conformal vector field are considered. Finally, the structure of ρ-Einstein solitons on some warped products space-times is investigated.

1. An introduction

Ricci soliton is crucial in the Ricci flow treatment. In [10, 12], the Ricci flow is defined on a Riemannian manifold (E, g) by an evolution equation for metrics $\{g(t)\}$ of the form

$$\partial_t g(t) = -2\text{Ric}, \tag{1.1}$$

where Ric is the Ricci curvature tensor. The initial metric g on E satisfies

$$\text{Ric} + \frac{1}{2}\mathcal{L}_\zeta g = \lambda g, \tag{1.2}$$

where ζ is a vector field on E, λ is a constant, and $\mathcal{L}_\zeta$ represents the Lie derivative in the direction of a vector field ζ on E. Manifolds admitting such structure are called Ricci soliton [13]. Hamilton first investigated the study of Ricci solitons as fixed points of the Ricci flow in the space of the metrics on E modulo diffeomorphisms and scaling [19]. A Ricci soliton is called shrinking (steady, or expanding) if $\lambda > 0$ ($\lambda = 0$, or $\lambda < 0$ respectively). If $\zeta = 0$ or is Killing, then the Ricci soliton is called a trivial Ricci soliton. If f is a smooth function and $\zeta = \nabla f$, then the Ricci soliton is called gradient, ζ is called the potential vector field, and f is called the potential function. In this case, equation (1.2) becomes

$$\text{Ric} + H^f = \lambda g, \tag{1.3}$$

where H^f is the Hessian tensor. For different reasons and in distinct spaces, Ricci solitons have been remarkably studied [5,6,21,24,28,30,31]. In [32], it is shown that a complete Ricci soliton is gradient. Gradient Ricci solitons are basic generalizations of Einstein manifolds [4]. If λ is a smooth function, then we say that (E, g) is a

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nearly Ricci soliton manifold [2,3,33]. A generalization of Einstein soliton has been deduced by considering the Ricci-Bourguignon flows [7–9]

$$\partial_t g(t) = -2\left(\mathrm{Ric} - \rho R g\right). \tag{1.4}$$

These manifolds are called ρ-Einstein solitons and are defined as follows. Let (E, g) be a pseudo-Riemannian manifold, and let $\lambda, \rho \in \mathbb{R}$, $\rho \neq 0$ and $\zeta \in \mathfrak{X}(E)$. Then (E, g, ζ, λ) is called a ρ-Einstein soliton if

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_\zeta g = \lambda g + \rho R g. \tag{1.5}$$

Likewise, if a smooth function $f : E \to \mathbb{R}$ exists such that $\zeta = \nabla f$, then a ρ-Einstein soliton (E, g, ζ, ρ) is gradient and denoted by (E, g, f, ρ). In this case, equation (1.5) becomes

$$\mathrm{Ric} + Hess\,(f) = \lambda g + \rho R g. \tag{1.6}$$

As usual, a ρ-Einstein soliton is called steady, shrinking or expanding on whether λ has zero, positive or negative values. The function f is called a ρ-Einstein potential of the gradient ρ-Einstein soliton. Later, this perception was circulated in many instructions, such as m-quasi Einstein manifolds [20], Ricci-Bourguignon almost solitons [14], (E, ρ)-quasi-Einstein manifolds [22], etc. Huang got a sufficient condition for a compact gradient shrinking ρ-Einstein soliton to be isometric to a quotient of the round sphere S^n in [23]. Moreover, Mondal and Shaikh proved that a compact gradient ρ-Einstein soliton with a non−trivial conformal vector field ∇f, is isometric to the Euclidean sphere S^n in [27]. Recently, in [14] Dwivedi demonstrated other isometric theories of gradient Ricci-Bourguignon soliton. In [40], the authors investigated gradient ρ-Einstein soliton on Kenmotsu manifold. Some curvature conditions on compact gradient ρ-Einstein soliton M are given in [34] to guarantee that M is isometric to the Euclidean sphere. In contrast, an integral condition on a non-compact ρ-Einstein soliton M is given to ensure the vanishing of the scalar curvature. A splitting theorem of gradient ρ-Einstein soliton is given in [36]. Accordingly, many characterizations of gradient ρ-Einstein solitons are considered in [35]. The same study is recently extended to Sasakian manifolds in [29]. A study of the lower bound of the diameter of a compact gradient ρ-Einstein soliton is given in [37].

As far as we know, no research has been done on such a structure on warped product manifolds. Research problems in this regard from the point of view of warped product manifolds (WPM)s can be summarized into two paths:

(1) Under what conditions does a WPM become a ρ-Einstein soliton or a gradient ρ-Einstein soliton?
(2) What does a factor of a ρ-Einstein soliton WPM or a gradient ρ-Einstein soliton WPM inherit?

To address these problems, first we proved many results on ρ-Einstein soliton. Then, we investigate necessary and sufficient conditions on (gradient) ρ-Einstein soliton WPM in order to make its factors (gradient) ρ-Einstein soliton. Also, we study ρ-Einstein soliton on a WPM admitting a conformal vector field. Finally, we apply our results to GRW space-times and standard static space-times.

2. Preliminaries

2.1. **ρ-Einstein solitons on pseudo-Riemannian manifolds.** If ζ is a conformal vector field with conformal factor 2ω in a ρ-Einstein soliton (E, g, ζ, λ), then

$$\begin{aligned} \mathrm{Ric}\,(U,V) + \frac{1}{2}\mathcal{L}_\zeta g\,(U,V) &= \lambda g\,(U,V) + \rho R g\,(U,V) \\ \mathrm{Ric}\,(U,V) + \omega g\,(U,V) &= \lambda g\,(U,V) + \rho R g\,(U,V) \\ \mathrm{Ric}\,(U,V) &= (\lambda - \omega + \rho R)\, g\,(U,V)\,. \end{aligned} \tag{2.1}$$

By taking the trace over U, V, we get

$$\begin{aligned} \frac{R}{n} &= \lambda - \omega + \rho R \\ R &= \frac{(\lambda - \omega)\, n}{1 - n\rho}. \end{aligned} \tag{2.2}$$

Since the scalar curvature of Einstein manifolds is constant, the conformal factor is also constant, that is, ζ is homothetic. Moreover, $\lambda = \omega$ if $\rho = \frac{1}{n}$.

Proposition 1. *Assume that ζ is a conformal vector field on a ρ-Einstein soliton (E, g, ζ, λ) with factor 2ω. Then, ζ is homothetic, (E, g) is Einstein, and*

$$R = \frac{(\lambda - \omega)\, n}{1 - n\rho}.$$

where $\rho \neq \frac{1}{n}$. Moreover, $\lambda = \omega$ if $\rho = \frac{1}{n}$.

Corollary 1. *Assume that ζ is a Killing vector field on a ρ-Einstein soliton (E, g, ζ, λ), then*

$$R = \frac{n\lambda}{1 - n\rho}$$

where $\rho \neq \frac{1}{n}$. Moreover, (E, g, ζ, λ) is steady if $\rho = \frac{1}{n}$.

Conversely, assuming that (E, g) is an Einstein manifold, then

$$\begin{aligned} \frac{R}{n} g\,(U,V) + \frac{1}{2}\left(\mathcal{L}_\zeta g\right)(U,V) &= \lambda g\,(U,V) + \rho R g\,(U,V) \\ \left(\mathcal{L}_\zeta g\right)(U,V) &= \left(\lambda - \frac{R}{n} + \rho R\right) g\,(U,V) \end{aligned}$$

Therefore, ζ is a homothetic vector field on E.

Proposition 2. *In a ρ-Einstein soliton (E, g, ζ, λ), ζ is a homothetic vector field on E if (E, g) is Einstein. Furthermore, ζ is Killing if $\lambda = \left(\frac{1}{n} - \rho\right) R$.*

In local coordinates, a contraction of the defining equation implies

$$\begin{aligned} R_{ij} + \frac{1}{2}\left(\nabla_i \zeta_j + \nabla_j \zeta_i\right) &= \lambda g_{ij} + \rho R g_{ij} \\ \nabla_i \zeta^i &= n\lambda + (n\rho - 1)\, R \end{aligned}$$

Thus, the vector field ζ is divergence-free. The conservative laws in physics usually arise from the vanishing of the divergence of a tensor field. Here is a simple characterization of the vanishing of the divergence of ζ.

Corollary 2. *The vector field ζ in a ρ-Einstein soliton (E, g, ζ, λ) is divergence-free if and only if $n\lambda + (n\rho - 1)\, R = 0$.*

It is also known that the flow lines of a divergence-free vector field are volume-preserving diffeomorphisms [1, Chapter 3]. This discussion leads to the following result.

Theorem 1. *The flow lines of the vector field ζ in a ρ-Einstein soliton (E, g, ζ, λ) are volume-preserving diffeomorphisms if and only if $n\lambda + (n\rho - 1) R = 0$.*

2.2. **Warped product manifolds.** Let (E_i, g_i, D^i), $i = 1, 2$ denote two n_i-dimensional C^∞ pseudo-Riemannian manifolds equipped with metric tensors g_i where D^i is the Levi-Civita connection of the metric g_i for $i = 1, 2$. Let $f_1 : E_1 \to (0, \infty)$ be a smooth positive real-valued function. A WPM, denoted by $E = E_1 \times_f E_2$, is the product manifold $E_1 \times E_2$ equipped with the metric tensor $g = g_1 \oplus f^2 g_2$ (For more details the reader is referred to [15, 17, 25, 38, 39] and references therein). Let $E = E_1 \times_f E_2$ be a pseudo-Riemannian WPM and $U_i, V_i \in \mathfrak{X}(E_i)$ for all $i = 1, 2$. Then, the Ricci tensor Ric of E is given by

(1) $\mathrm{Ric}(U_1, V_1) = \mathrm{Ric}^1(U_1, V_1) - \frac{n_2}{f} H^f(U_1, V_1)$,
(2) $\mathrm{Ric}(U_1, U_2) = 0$,
(3) $\mathrm{Ric}(U_2, V_2) = \mathrm{Ric}^2(U_2, V_2) - f^\circ g_2(U_2, V_2)$, where $f^\circ = f\Delta f + (n_2 - 1)\|\nabla f\|_1^2$, and Δ is the Laplacian on E_1.

The scalar curvature a WPM satisfies

$$R = R_1 + \frac{1}{f^2} R_2 - 2n\frac{\Delta f}{f} - n(n-1)\frac{1}{f^2} g_1(\nabla f, \nabla f). \tag{2.3}$$

Lemma 1. *[38] In a WPM $E_1 \times_f E_2$, the Lie derivative with respect to a vector field $\zeta = \zeta_1 + \zeta_2$ satisfies*

$$\mathcal{L}_\zeta g(U, V) = \left(\mathcal{L}^1_{\zeta_1} g_1\right)(U_1, V_1) + f^2\left(\mathcal{L}^2_{\zeta_2} g_2\right)(U_2, V_2) + 2f\zeta_1(f) g_2(U_2, V_2), \tag{2.4}$$

for any vector fields $U = U_1 + U_2$, $V = V_1 + V_2$, where $\mathcal{L}^i_{\zeta_i}$ is the Lie derivative on E_i with respect to ζ_i, for $i = 1, 2$.

3. ρ-Einstein solitons structure on $WPMs$

In this section, we investigate ρ-Einstein soliton structure on $WPMs$. For the rest of this work, let $E = E_1 \times_f E_2$ be a WPM with warping function f and let $g = g_1 \oplus f^2 g_2$. Also, let $\zeta = \zeta_1 + \zeta_2$ be a vector field on E. Let (E, g, ζ, λ), be a ρ-Einstein soliton, that is,

$$\mathrm{Ric}(U, V) + \frac{1}{2}\mathcal{L}_\zeta g(U, V) = \lambda g(U, V) + \rho R g(U, V). \tag{3.1}$$

Thus, for any vector fields $U = U_1 + U_2$, $V = V_1 + V_2$ and $\zeta = \zeta_1 + \zeta_2$ on $E = E_1 \times_f E_2$, Lemma 1 implies

$$\begin{aligned}
&\mathrm{Ric}^1(U_1, V_1) - \frac{n_2}{f} H^f(U_1, V_1) + \mathrm{Ric}^2(U_2, V_2) - f^\circ g_2(U_2, V_2) \\
&+\frac{1}{2}\left(\mathcal{L}^1_{\zeta_1} g_1\right)(U_1, V_1) + \frac{1}{2} f^2\left(\mathcal{L}^2_{\zeta_2} g_2\right)(U_2, V_2) + f\zeta_1(f) g_2(U_2, V_2) \\
=\ & \lambda g_1(U_1, V_1) + \lambda f^2 g_2(U_2, V_2) + \rho R g_1(U_1, V_1) + \rho R f^2 g_2(U_2, V_2).
\end{aligned} \tag{3.2}$$

Let $U=U_1$, $V=V_1$ and $H^f=\sigma g$, then

$$\begin{aligned} \operatorname{Ric}^1(U_1,V_1)+\frac{1}{2}\left(\mathcal{L}^1_{\zeta_1}g_1\right)(U_1,V_1) &= \lambda_1 g_1(U_1,V_1) \\ &\quad +\left[-\lambda_1+\lambda+\frac{n_2}{f}\sigma+\rho R\right]g_1(U_1,V_1) \qquad (3.3)\\ &= \lambda_1 g_1(U_1,V_1)+\rho_1 R_1 g_1(U_1,V_1). \qquad (3.4)\end{aligned}$$

Then, $(E_1,g_1,\zeta_1,\lambda_1)$ is a ρ_1-Einstein soliton, where

$$\rho_1 R_1+\lambda_1=\rho R+\frac{n_2}{f}\sigma+\lambda.$$

Now, let $U=U_2$ and $V=V_2$, then

$$\begin{aligned} &\operatorname{Ric}^2(U_2,V_2)-f^\circ g_2(U_2,V_2)\\ &+\frac{1}{2}f^2\left(\mathcal{L}^2_{\zeta_2}g_2\right)(U_2,V_2)+f\zeta_1(f)\,g_2(U_2,V_2)\\ =\;& \lambda f^2 g_2(U_2,V_2)+\rho R f^2 g_2(U_2,V_2).\end{aligned}$$

Thus,

$$\begin{aligned} &\operatorname{Ric}^2(U_2,V_2)+\frac{1}{2}f^2\left(\mathcal{L}^2_{\zeta_2}g_2\right)(U_2,V_2)\\ =\;& \left[\lambda f^2+f^\circ-f\zeta_1(f)+\rho R f^2\right]g_2(U_2,V_2)\\ =\;& \lambda_2 g_2(U_2,V_2)+\left[-\lambda_2+\lambda f^2+f^\circ-f\zeta_1(f)+\rho R f^2\right]g_2(U_2,V_2)\\ =\;& \lambda_2 g_2(U_2,V_2)+\rho_2 R_2 g_2(U_2,V_2). \qquad (3.5)\end{aligned}$$

Then, $\left(E_2,g_2,f^2\zeta_2,\lambda_2\right)$ is a ρ_2-Einstein soliton, where

$$\rho_2 R_2+\lambda_2=\rho R f^2+\lambda f^2+f^\circ-f\zeta_1(f). \qquad (3.6)$$

Theorem 2. *Let (E,g,ζ,λ,ρ) be a ρ-Einstein soliton. Then,*

1. *$(E_1,g_1,\zeta_1,\lambda_1)$ is a ρ_1-Einstein soliton if $H^f=\sigma g$ where*
$$\rho_1 R_1+\lambda_1=\rho R+\frac{n_2}{f}\sigma+\lambda.$$
2. *$\left(E_2,g_2,f^2\zeta_2,\lambda_2\right)$ is a ρ_2-Einstein soliton, where*
$$\rho_2 R_2+\lambda_2=\rho R f^2+\lambda f^2+f^\circ-f\zeta_1(f).$$

Let (E_1,g_1) and (E_2,g_2) be two Einstein manifolds with factors μ_1 and μ_2 respectively, and let $H^f=\sigma g$. Then Equation (3.2) becomes

$$\begin{aligned} &\mu_1 g_1(U_1,V_1)+\mu_2 g_2(U_2,V_2)-\frac{n_2}{f}\sigma g_1(U_1,V_1)-f^\circ g_2(U_2,V_2)\\ &+\frac{1}{2}\left(\mathcal{L}^1_{\zeta_1}g_1\right)(U_1,V_1)+\frac{1}{2}f^2\left(\mathcal{L}^2_{\zeta_2}g_2\right)(U_2,V_2)+f\zeta_1(f)\,g_2(U_2,V_2)\\ =\;& \lambda g_1(U_1,V_1)+\lambda f^2 g_2(U_2,V_2)+\rho R g_1(U_1,V_1)+\rho R f^2 g_2(U_2,V_2).\end{aligned}$$

Thus,

$$\left(\mathcal{L}^1_{\zeta_1}g_1\right)(U_1,V_1) = 2\left[\lambda+\frac{n_2}{f}\sigma-\mu_1+\rho R\right]g_1(U_1,V_1), \qquad (3.7)$$

$$\left(\mathcal{L}^2_{\zeta_2}g_2\right)(U_2,V_2) = \frac{2}{f^2}\left[f^\circ-\mu_2-f\zeta_1(f)+\lambda f^2+\rho R f^2\right]g_2(U_2,V_2). \qquad (3.8)$$

That is, ζ_1 and ζ_2 are conformal vector fields on E_1 and E_2.

Theorem 3. *In a ρ-Einstein soliton (E, g, ζ, λ), $E = E_1 \times_f E_2$,*

(1) *ζ_1 is conformal vector field on E_1 if $H^f = \sigma g$ and (E_1, g_1) is Einstein, and*
(2) *ζ_2 is conformal vector field on E_2 if (E_2, g_2) is Einstein.*

The symmetry assumptions induced by Killing vector fields, denoted by KVF, are widely used in general relativity to gain a better understanding of the relationship between matter and the geometry of a space-time. In this case, the metric tensor does not change along the flow lines of a KVF. Such symmetry is measured by the number of independent KVFs. Manifolds of constant curvature admit the maximum number of independent KVFs. Similarly, conformal vector fields, denoted by CVF, play a crucial role in the study of space-time physics. The flow lines of a CVF are conformal transformations of the ambient space. Thus, the existence and characterization of CVFs in pseudo-Riemannian manifolds are essential and are extensively discussed by both mathematicians and physicists.

Now, assume that ζ is a conformal vector field on E, i.e., $\mathcal{L}_\zeta g = 2\omega g$ for some scalar function ω, then ω is constant and

$$\mathrm{Ric}(U, V) = (\lambda - \omega + \rho R)\, g(U, V). \tag{3.9}$$

This equation implies

$$\begin{aligned} & \mathrm{Ric}^1(U_1, V_1) - \frac{n_2}{f} H^f(U_1, V_1) + \mathrm{Ric}^2(U_2, V_2) - f^\circ g_2(U_2, V_2) \\ = \; & [\lambda - \omega + \rho R]\, g_1(U_1, V_1) + [\lambda - \omega + \rho R]\, f^2 g_2(U_2, V_2). \end{aligned} \tag{3.10}$$

If $H^f = \sigma g$, then

$$\begin{aligned} \mathrm{Ric}^1(U_1, V_1) &= \left[\lambda - \omega + \rho R + \frac{n_2}{f}\sigma\right] g_1(U_1, V_1), \qquad (3.11)\\ \mathrm{Ric}^2(U_2, V_2) &= \left[f^\circ + \lambda f^2 - \omega f^2 + \rho R f^2\right] g_2(U_2, V_2). \end{aligned}$$

That is, both the base and fibre manifolds are Einstein.

Theorem 4. *In a ρ-Einstein soliton (E, g, ζ, λ), $E = E_1 \times_f E_2$ admitting a conformal vector field $\zeta = \zeta_1 + \zeta_2$,*

(1) *(E_1, g_1) is Einstein if $H^f = \sigma g$, and*
(2) *(E_2, g_2) is Einstein.*

The condition $H^f = \sigma g$ is equivalent to ∇f is a concircular vector field. Equation (3.2) yields

$$\begin{aligned} & \mathrm{Ric}^1(U_1, V_1) - \frac{n_2}{f} H^f(U_1, V_1) + \frac{1}{2}\left(\mathcal{L}^1_{\zeta_1} g_1\right)(U_1, V_1) \\ = \; & \lambda g_1(U_1, V_1) + \rho R g_1(U_1, V_1). \end{aligned}$$

Suppose that ∇f is a concircular vector field with factor γ, i.e., $D_{U_1} \nabla f = \gamma U_1$, we get

$$\begin{aligned} & \mathrm{Ric}^1(U_1, V_1) + \frac{1}{2}\left(\mathcal{L}^1_{\zeta_1} g_1\right)(U_1, V_1) \\ = \; & \lambda g_1(U_1, V_1) + \left[\frac{\gamma n_2}{f} + \rho R\right] g_1(U_1, V_1) \qquad (3.12)\\ = \; & \lambda_1 g_1(U_1, V_1) + \left[-\lambda_1 + \lambda + \frac{\gamma n_2}{f} + \rho R\right] g_1(U_1, V_1) \\ = \; & \lambda_1 g_1(U_1, V_1) + \rho_1 R_1 g_1(U_1, V_1). \qquad (3.13) \end{aligned}$$

Then, $(E_1, g_1, \zeta_1, \lambda_1)$ is a ρ_1-Einstein soliton where

$$\rho_1 R_1 + \lambda_1 = \frac{\gamma n_2}{f} + \rho R. \tag{3.14}$$

Corollary 3. *In a ρ-Einstein soliton $(E, g, \zeta, \lambda, \rho)$, assume that ∇f is a concircular vector field with factor γ, then $(E_1, g_1, \zeta_1, \lambda_1)$ is a ρ_1-Einstein soliton where*

$$\rho_1 R_1 + \lambda_1 = \frac{\gamma n_2}{f} + \rho R. \tag{3.15}$$

Bang-Yen Chen proved that a Riemannian manifold admitting a concircular vector field is locally a warped product of the form $I \times_\varphi \bar{E}_1$ [11]. Thus, the aforementioned warped product manifold becomes a sequential warped product manifold [16].

From Lemma 1, it is clear that ζ_1, ζ_2 are conformal vector fields on E_1, E_2 with conformal factors η_1, η_2, respectively. Then, by employing equation 3.11 we get

$$\begin{aligned}
\mathcal{L}^1_{\zeta_1} \mathrm{Ric}^1(U_1, V_1) &= \left[\frac{n_2}{f}\sigma + \lambda - \omega + \rho R\right] \mathcal{L}^1_{\zeta_1} g_1(U_1, V_1) \\
&\quad + \zeta_1 \left(\frac{n_2}{f}\sigma + \lambda - \omega + \rho R\right) g_1(U_1, V_1). \\
\mathcal{L}^1_{\zeta_1} \mathrm{Ric}^1(U_1, V_1) &= \begin{bmatrix} \left(\frac{n_2}{f}\sigma + \lambda - \omega + \rho R\right)\eta_1 \\ + \zeta_1 \left(\frac{n_2}{f}\sigma + \lambda - \omega + \rho R\right) \end{bmatrix} g_1(U_1, V_1) \\
&= \varphi_1 g_1(U_1, V_1).
\end{aligned}$$

where

$$\varphi_1 = \left[\frac{n_2}{f}\sigma + \lambda - \omega + \rho R\right]\eta_1 + \zeta_1\left(\frac{n_2}{f}\sigma + \lambda - \omega + \rho R\right). \tag{3.16}$$

Also,

$$\begin{aligned}
\mathcal{L}^2_{\zeta_2} \mathrm{Ric}^2(U_2, V_2) &= \left[\left(\frac{f^\circ}{f^2} + \lambda - \omega\right) f^2 + \rho R f^2\right] \mathcal{L}^2_{\zeta_2} g_2(U_2, V_2) \\
\mathcal{L}^2_{\zeta_2} \mathrm{Ric}^2(U_2, V_2) &= f^2 \left[\frac{f^\circ}{f^2} + \lambda - \omega + \rho R\right] \eta_2 g_2(U_2, V_2) \\
&= \varphi_2 g_2(U_2, V_2) \qquad (3.17)
\end{aligned}$$

where

$$\varphi_2 = f^2 \left[\frac{f^\circ}{f^2} + \lambda - \omega + \rho R\right] \eta_2. \tag{3.18}$$

Theorem 5. *In a ρ-Einstein soliton (E, g, ζ, λ) admitting a conformal vector field ζ with factor ω,*

(1) $\mathcal{L}^1_{\zeta_1} \mathrm{Ric}^1(U_1, V_1) = \varphi_1 g_1(U_1, V_1)$ *if* $H^f = \sigma g$, *where*

$$\varphi_1 = \left[\frac{n_2}{f}\sigma + \lambda - \omega + \rho R\right]\eta_1 + \zeta_1\left(\frac{n_2}{f}\sigma + \lambda - \omega + \rho R\right), \tag{3.19}$$

(2) $\mathcal{L}^2_{\zeta_2} \mathrm{Ric}^2(U_2, V_2) = \varphi_2 g_2(U_2, V_2)$, *where*

$$\varphi_2 = f^2 \left[\frac{f^\circ}{f^2} + \lambda - \omega + \rho R\right] \eta_2. \tag{3.20}$$

The Killing vector fields provide the isometries of space-time whereas the symmetry of the energy-momentum tensor is given by the Ricci collineation. A vector field ζ represents a Ricci collineation if the Ricci tensor is invariant under the Lie dragging through flow lines of ζ. The foregoing conclusion establishes the shape of the Lie derivative of the Ricci tensor with regard to the fields ζ_i, on M_i, $i = 1, 2$.

Let $(E, g, \zeta, \lambda, \rho)$ be a gradient ρ-Einstein soliton with $\zeta = \nabla u$, then

$$\mathrm{Ric} + H^u = \lambda g + \rho R g.$$

Thus,

$$\begin{aligned} & \mathrm{Ric}\left(U_1+U_2, V_1+V_2\right) + H^u\left(U_1+U_2, V_1+V_2\right) \\ = & \lambda g\left(U_1+U_2, V_1+V_2\right) + \rho R g\left(U_1+U_2, V_1+V_2\right). \end{aligned}$$

Let $U = U_1, V = V_1$

$$\begin{aligned} & \mathrm{Ric}^1\left(U_1, V_1\right) - \frac{n_2}{f} H^f\left(U_1, V_1\right) + H_1^{u_1}\left(U_1, V_1\right) \\ = & \lambda g_1\left(U_1, V_1\right) + \rho R g_1\left(U_1, V_1\right) \end{aligned}$$

$$\begin{aligned} & \mathrm{Ric}^1\left(U_1, V_1\right) + H_1^{\phi_1}\left(U_1, V_1\right) \\ = & \lambda_1 g_1\left(U_1, V_1\right) + \left(-\lambda_1 + \lambda + \rho R\right) g_1\left(U_1, V_1\right) \\ = & \lambda_1 g_1\left(U_1, V_1\right) + \rho_1 R_1 g_1\left(U_1, V_1\right), \end{aligned}$$

where $\phi_1 = u_1 - u_2 \ln f$ and $u_1 = u$ at a fixed point of E_2. Then, $(E_1, g_1, \zeta_1, \rho_1)$ is a gradient ρ_1-Einstein soliton where

$$\rho_1 R_1 + \lambda_1 = \lambda + \rho R.$$

Now, let $U = U_2, V = V_2$, then

$$\begin{aligned} & \mathrm{Ric}^2\left(U_2, V_2\right) - f^{\circ} g_2\left(U_2, V_2\right) + H_2^{\phi_2}\left(U_2, V_2\right) \\ = & \lambda f^2 g_2\left(U_2, V_2\right) + \rho R f^2 g_2\left(U_2, V_2\right). \end{aligned}$$

This yields

$$\begin{aligned} & \mathrm{Ric}^2\left(U_2, V_2\right) + H_2^{\phi_2}\left(U_2, V_2\right) \\ = & \lambda_2 g_2\left(U_2, V_2\right) + \left(-\lambda_2 + \lambda f^2 + f^{\circ} + \rho R f^2\right) g_2\left(U_2, V_2\right) \\ = & \lambda_2 g_2\left(U_2, V_2\right) + \rho_2 R_2 g_2\left(U_2, V_2\right), \end{aligned}$$

where $u_2 = u$ at a fixed point of E_1. Then, $(E_2, g_2, \zeta_2, \rho_2)$ is a gradient ρ_2-Einstein soliton where

$$\rho_2 R_2 + \lambda_2 = \lambda f^2 + f^{\circ} + \rho R f^2.$$

Theorem 6. *In a gradient ρ-Einstein soliton (E, g, ζ, λ),*

(1) $(E_1, g_1, \zeta_1, \lambda_1)$ *is a gradient ρ_1-Einstein soliton where*

$$\rho_1 R_1 + \lambda_1 = \lambda + \rho R,$$

(2) $(E_2, g_2, \zeta_2, \lambda_2)$ *is a gradient ρ_2-Einstein soliton where*

$$\rho_2 R_2 + \lambda_2 = \lambda f^2 + f^{\circ} + \rho R f^2.$$

This theorem provides an inheritance property of the structure of the gradient ρ-Einstein soliton structure to factor manifolds of the warped product manifold.

3.1. **$\bar{\rho}$-Einstein solitons on a generalized Robertson-Walker space-times.** Let $\bar{E} = I \times_f E$ be a generalized Robertson-Walker space-time with metric $\bar{g} = -dt^2 \oplus f^2 g$. Then the Ricci curvature tensor Ric on E is

$$\begin{aligned}\bar{\text{Ric}}(\partial_t, \partial_t) &= -\frac{n\ddot{f}}{f}, \quad \bar{\text{Ric}}(U, \partial_t) = 0 \\ \bar{\text{Ric}}(U, V) &= \text{Ric}(U, V) - f^{\diamondsuit} g(U, V),\end{aligned}$$

where $f^{\diamondsuit} = -f\ddot{f} - (n-1)\dot{f}^2$, see [16, 18, 26].

Lemma 2. *Suppose that $h\partial_t, u\partial_t, v\partial_t \in \mathfrak{X}(I)$ and $\zeta, U, V \in \mathfrak{X}(E)$, then*

$$\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g}\left(\bar{U}, \bar{V}\right) = -2\dot{h}uv + f^2 \mathcal{L}_\zeta g(U, V) + 2hf\dot{f}g(U, V),$$

where $\bar{U} = u\partial_t + U, \bar{V} = v\partial_t + V$ and $\bar{\zeta} = h\partial_t + \zeta$.

Let $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$, $\bar{E} = I \times_f E$, be a $\bar{\rho}$-Einstein soliton GRW space-time. Then,

$$\bar{\text{Ric}}\left(\bar{U}, \bar{V}\right) + \frac{1}{2}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g}\left(\bar{U}, \bar{V}\right) = \bar{\lambda}\bar{g}\left(\bar{U}, \bar{V}\right) + \bar{\rho}\bar{R}\bar{g}\left(\bar{U}, \bar{V}\right),$$

where $\bar{U} = u\partial_t + U, \bar{V} = v\partial_t + V$ and $\bar{\zeta} = h\partial_t + \zeta$ are vector fields on $\bar{E}$. Thus,

$$\begin{aligned}&-\frac{n\ddot{f}}{f}uv + \text{Ric}(U, V) - f^{\diamondsuit} g(U, V) - \dot{h}uv + \frac{1}{2}f^2\mathcal{L}_\zeta g(U, V) + hf\dot{f}g(U, V) \\ = \;& -\bar{\lambda}uv + f^2\bar{\lambda}g(U, V) - \bar{\rho}\bar{R}uv + \bar{\rho}\bar{R}f^2 g(U, V).\end{aligned}$$

This yields

$$n\ddot{f} = f\left(\bar{\lambda} - \dot{h}\right) + \bar{\rho}\bar{R}f,$$

and

$$\begin{aligned}&\text{Ric}(U, V) + \frac{1}{2}f^2\mathcal{L}_\zeta g(U, V) \\ = \;& \bar{\lambda}f^2 g(U, V) + \bar{\rho}\bar{R}f^2 g(U, V) + f^{\diamondsuit} g(U, V) - hf\dot{f}g(U, V).\end{aligned}$$

Thus, $(E, g, f^2\zeta, \rho)$ is a ρ-Einstein soliton, where

$$\rho R + \lambda = \left(\bar{\lambda} + \bar{\rho}\bar{R}\right)f^2 + f^{\diamondsuit} - hf\dot{f}.$$

Theorem 7. *In a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$, where $\bar{E} = I \times_f E$ is a generalized Robertson-Walker space-time, it is*

(1) $n\ddot{f} = f\left(\bar{\lambda} - \dot{h}\right) + \bar{\rho}\bar{R}f$,
(2) $(E, g, f^2\zeta, \lambda)$ is a ρ-Einstein soliton, where
$$\rho R + \lambda = \left(\bar{\lambda} + \bar{\rho}\bar{R}\right)f^2.$$

In a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$, where $\bar{E} = I \times_f E$ is a generalized Robertson-Walker space-time and $\bar{\zeta} = h\partial_t + \zeta$ is a conformal vector field on $\bar{E}$, i.e., $\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g} = \bar{\omega}\bar{g}$, and $\bar{\omega}$ is constant (see Section 2), then

$$\bar{\text{Ric}}\left(\bar{U}, \bar{V}\right) = \left(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\right)\bar{g}\left(\bar{U}, \bar{V}\right).$$

Thus,

$$\begin{aligned}&-\frac{n\ddot{f}}{f}uv + \text{Ric}(U, V) - f^{\diamondsuit} g(U, V) \\ = \;& -\left(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\right)uv + \left(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\right)f^2 g(U, V).\end{aligned}$$

Thus,

$$\frac{n\ddot{f}}{f} = \bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}. \tag{3.21}$$

$$\mathrm{Ric}\,(U, V) = \left[f^{\diamondsuit} + \left(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\right) f^2\right] g\,(U, V)\,. \tag{3.22}$$

By using equations (3.21) we get

$$\mathrm{Ric}\,(U, V) = \left[(n-1)\left(f\ddot{f} - \dot{f}^2\right)\right] g\,(U, V)\,.$$

Therefore, (E, g) is an Einstein manifold with factor $\mu = (n-1)\left(f\ddot{f} - \dot{f}^2\right)$.

Theorem 8. *In a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$ admitting a conformal vector field $\bar{\zeta} = h\partial_t + \zeta$, where $\bar{E} = I \times_f E$ is a generalized Robertson-Walker space-time, (E, g) is an Einstein manifold with factor $\mu = (n-1)\left(f\ddot{f} - \dot{f}^2\right)$.*

From Lemma 2, we get ζ is a conformal vector field on E with conformal factor η. Then, by using theorem 8, we get

$$\begin{aligned}
\mathcal{L}_{\zeta}\mathrm{Ric}\,(U, V) &= \left[(n-1)\left(f\ddot{f} - \dot{f}^2\right)\right] \mathcal{L}_{\zeta} g\,(U, V) \\
&= (n-1)\left(f\ddot{f} - \dot{f}^2\right) \eta g\,(U, V) \\
&= \varphi g\,(U, V)\,,
\end{aligned}$$

where

$$\varphi = (n-1)\left(f\ddot{f} - \dot{f}^2\right) \eta.$$

Theorem 9. *In a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$ admitting a conformal vector field $\bar{\zeta} = h\partial_t + \zeta$, where $\bar{E} = I \times_f E$ is a generalized Robertson-Walker space-time,*

$$\mathcal{L}_{\zeta}\mathrm{Ric}\,(U, V) = \varphi g\,(U, V)\,,$$

where

$$\varphi = (n-1)\left(f\ddot{f} - \dot{f}^2\right) \eta.$$

In a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$, where $\bar{E} = I \times_f E$ is a generalized Robertson-Walker space-time, it is

$$\bar{\mathrm{R}}\mathrm{ic}\left(\bar{U}, \bar{V}\right) + \frac{1}{2}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g}\left(\bar{U}, \bar{V}\right) = \bar{\lambda}\bar{g}\left(\bar{U}, \bar{V}\right) + \bar{\rho}\bar{R}\bar{g}\left(\bar{U}, \bar{V}\right).$$

Assume that (E, g) is Einstein, then for any vector fields $\bar{U} = U, \bar{V} = V$ and $\bar{\zeta} = h\partial_t + \zeta$ we have get

$$\begin{aligned}
\mathcal{L}_{\zeta} g\,(U, V) &= 2\left[\frac{1}{f^2}\left(-\mu + f^{\diamondsuit} - hf\dot{f} + f^2\bar{\lambda}\right) + \bar{\rho}\bar{R}\right] g\,(U, V) \\
&= \eta g\,(U, V)\,.
\end{aligned}$$

Then, ζ is a conformal vector field on E with conformal factor η where

$$\eta = 2\left[\frac{1}{f^2}\left(-\mu + f^{\diamondsuit} - hf\dot{f} + f^2\bar{\lambda}\right) + \bar{\rho}\bar{R}\right].$$

Theorem 10. *In a $\bar{\rho}$-Einstein soliton $\left(\bar{E},\bar{g},\bar{\zeta},\bar{\lambda}\right)$, where $\bar{E}=I\times_f E$ is a generalized Robertson-Walker space-time, ζ is a conformal vector field on E if (E,g) is Einstein manifold with conformal factor η where*

$$\eta = 2\left[\frac{1}{f^2}\left(-\mu + f^{\diamondsuit} - hf\dot{f} + f^2\bar{\lambda}\right) + \bar{\rho}\bar{R}\right].$$

3.2. **$\bar{\rho}$-Einstein solitons on a standard static space-times.** A standard static space-time (also called f-associated SSST) is a Lorentzian warped product manifold $\bar{E}=I_f\times E$ furnished with the metric $\bar{g}=-f^2dt^2\oplus g$. The Ricci curvature tensor Ric on E is

$$\begin{aligned}\bar{\mathrm{Ric}}\left(\partial_t,\partial_t\right) &= f\Delta f \quad \bar{\mathrm{Ric}}\left(U,\partial_t\right)=0\\ \bar{\mathrm{Ric}}\left(U,V\right) &= \mathrm{Ric}\left(U,V\right)-\frac{1}{f}H^f\left(U,V\right),\end{aligned}$$

where Δf denotes the Laplacian of f on E. This space-time is a generalization of some notable classical space-times. The Einstein static universe and Minkowski space-time are good examples of standard static space-times [4].

Lemma 3. *Suppose that $h\partial_t, u\partial_t, v\partial_t\in\mathfrak{X}(I)$ and $\zeta,U,V\in\mathfrak{X}(E)$, then*

$$\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g}\left(\bar{U},\bar{V}\right)=\mathcal{L}_{\zeta}g\left(U,V\right)-2uvf^2\left(\dot{h}+\zeta\left(\ln f\right)\right),$$

where $\bar{U}=u\partial_t+U,\bar{V}=v\partial_t+V$ and $\bar{\zeta}=h\partial_t+\zeta$.

Let $\bar{E}=I_f\times E$ be a $\bar{\rho}$-Einstein soliton $\left(\bar{E},\bar{g},\bar{\zeta},\bar{\lambda}\right)$, then

$$\bar{\mathrm{Ric}}\left(\bar{U},\bar{V}\right)+\frac{1}{2}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g}\left(\bar{U},\bar{V}\right)=\bar{\lambda}\bar{g}\left(\bar{U},\bar{V}\right)+\bar{\rho}\bar{R}\bar{g}\left(\bar{U},\bar{V}\right),$$

where $\bar{U}=u\partial_t+U,\bar{V}=v\partial_t+V$ and $\bar{\zeta}=h\partial_t+\zeta$ are vector fields on $\bar{E}$. Then,

$$-\Delta f+f\dot{h}+\zeta\left(f\right)=\left[\bar{\lambda}+\bar{\rho}\bar{R}\right]f,$$

and

$$\begin{aligned}&\mathrm{Ric}\left(U,V\right)+\frac{1}{2}\mathcal{L}_{\zeta}g\left(U,V\right)\\ =\ &\bar{\lambda}g\left(U,V\right)+\bar{\rho}\bar{R}g\left(U,V\right)+\frac{1}{f}H^f\left(U,V\right).\end{aligned}$$

Suppose that $H^f\left(U,V\right)=\sigma g$, then

$$\mathrm{Ric}\left(U,V\right)+\frac{1}{2}\mathcal{L}_{\zeta}g\left(U,V\right)=\lambda g\left(U,V\right)+\rho Rg\left(U,V\right),$$

where

$$\rho R+\lambda=\bar{\lambda}+\frac{\sigma}{f}+\bar{\rho}\bar{R}.$$

Theorem 11. *If $H^f\left(U,V\right)=\sigma g$ in a $\bar{\rho}$-Einstein soliton $\left(\bar{E},\bar{g},\bar{\zeta},\bar{\lambda}\right)$ where $\bar{E}=I_f\times E$ is a standard static space-time, then (E,g,ζ,λ) is a ρ-Einstein soliton, where*

$$\rho R+\lambda=\bar{\lambda}+\frac{\sigma}{f}+\bar{\rho}\bar{R}.$$

The condition $H^f = \sigma g$ is equivalent to ∇f is a concircular vector field with factor γ, i.e., $D_U \nabla f = \gamma U$. Now, one gets

$$\begin{aligned} & \mathrm{Ric}\,(U,V) - \frac{\gamma}{f} g\,(U,V) + \frac{1}{2}\mathcal{L}_\zeta g\,(U,V) \\ = \; & \lambda g\,(U,V) + \left(-\lambda + \bar{\lambda} + \frac{\gamma}{f} + \bar{\rho}\bar{R}\right) g\,(U,V) \\ = \; & \lambda g\,(U,V) + \rho R g\,(U,V)\,. \end{aligned}$$

Then, (E, g) is an ρ-Einstein soliton where

$$\rho R + \lambda = \bar{\lambda} + \frac{\gamma}{f} + \bar{\rho}\bar{R}$$

Corollary 4. *If ∇f is a concircular vector field with factor σ on a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$ where $\bar{E} = I_f \times E$ is a standard static space-time, then (E, g, ζ, λ) is an ρ-Einstein soliton, where*

$$\rho R + \lambda = \bar{\lambda} + \frac{\gamma}{f} + \bar{\rho}\bar{R}.$$

Now, assume that $\bar{\zeta} = h\partial_t + \zeta$ is a conformal vector field on $\bar{E}$, i.e., $\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g} = \omega \bar{g}$, then

$$\bar{\mathrm{R}}\mathrm{ic}\left(\bar{U}, \bar{V}\right) = \left(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\right) \bar{g}\left(\bar{U}, \bar{V}\right).$$

Then

$$-\frac{\Delta f}{f} = \bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}. \tag{3.23}$$

Also,

$$\mathrm{Ric}\,(U,V) - \frac{1}{f} H^f\,(U,V) = \left(\bar{\lambda} - \bar{\omega} + \bar{\rho}\bar{R}\right) g\,(U,V)\,.$$

If $H^f\,(U,V) = \sigma g$, then by using equation(3.23) we get

$$\mathrm{Ric}\,(U,V) = \frac{1}{f}\left(\sigma - \Delta f\right) g\,(U,V)\,.$$

Thus, (E, g) is an Einstein manifold with factor $\mu = \frac{1}{f}\left(\sigma - \Delta f\right)$.

Theorem 12. *If $\bar{\zeta} = h\partial_t + \zeta$ is a conformal vector field on a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$ where $\bar{E} = I_f \times E$ is a standard static space-time and $H^f\,(U,V) = \sigma g$, then (E, g) is an Einstein manifold with factor $\mu = \frac{1}{f}\left(\sigma - \Delta f\right)$.*

From Lemma 3, we get ζ is a conformal vector field on E with conformal factor η. Then, by using theorem 12, we get

$$\mathcal{L}_\zeta \mathrm{Ric}\,(U,V) = \frac{1}{f}\left(\sigma - \Delta f\right) \mathcal{L}_\zeta g\,(U,V)\,.$$

Since $\bar{\zeta} = h\partial_t + \zeta$ is a conformal vector field on $\bar{E}$, ζ is a conformal vector field on E with conformal factor η. Thus

$$\mathcal{L}_\zeta \mathrm{Ric}\,(U,V) = \frac{1}{f}\left(\sigma - \Delta f\right) \eta g\,(U,V) = \varphi g\,(U,V)\,,$$

where

$$\varphi = \frac{1}{f}\left(\sigma - \Delta f\right) \eta.$$

Theorem 13. *If $\bar{\zeta} = h\partial_t + \zeta$ is a conformal vector field on a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$ where $\bar{E} = I_f \times E$ is a standard static space-time, then*

$$\mathcal{L}_\zeta \mathrm{Ric}\,(U, V) = \varphi g\,(U, V)\,,$$

where

$$\varphi = \frac{1}{f}\,(\sigma - \Delta f)\,\eta.$$

In a $\bar{\rho}$-Einstein soliton standard static space-time $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}, \rho\right)$, it is

$$\bar{\mathrm{Ric}}\left(\bar{U}, \bar{V}\right) + \frac{1}{2}\bar{\mathcal{L}}_{\bar{\zeta}}\bar{g}\left(\bar{U}, \bar{V}\right) = \bar{\lambda}\bar{g}\left(\bar{U}, \bar{V}\right) + \bar{\rho}\bar{R}\bar{g}\left(\bar{U}, \bar{V}\right).$$

Assume that (E, g) is Einstein manifold and $H^f\,(U, V) = \sigma g$, then

$$\mathcal{L}_\zeta g\,(U, V) = 2\left[\frac{\sigma}{f} - \mu + \bar{\lambda} + \bar{\rho}\bar{R}\right] g\,(U, V)\,.$$

Thus, ζ is a conformal vector field on E.

Theorem 14. *In a $\bar{\rho}$-Einstein soliton $\left(\bar{E}, \bar{g}, \bar{\zeta}, \bar{\lambda}\right)$ where $\bar{E} = I_f \times E$ is a standard static space-time, assume that (E, g) is Einstein manifold and $H^f\,(U, V) = \sigma g$, then ζ is a conformal vector field on E.*

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