Research Program

1 Scientific Background

Quivers without loops and 2-cycles (known as *cluster quivers*) form an important ingredient in the theory of cluster algebras by Fomin and Zelevinsky [14]. The notion of *quiver mutation*, which is the process of creating a new cluster quiver from a given one at a prescribed vertex, plays a central role in the theory.

A quiver can be enriched by an algebraic datum in the form of a *potential*. Roughly speaking, a potential is a linear combination of cycles in the (complete) path algebra of the quiver. A systematic study of quivers with potentials and their mutations has been initiated in the work [11]. Whereas in quiver mutation the resulting quiver has no 2-cycles by construction, this may not be the case for mutation of quivers with potentials. We say that a potential is *non-degenerate* if for any sequence of mutations, the resulting quiver does not have any 2-cycles.

Given a quiver with potential, one can construct several algebraic objects (e.g. the Ginzburg dg-algebra, the Jacobian algebra, a cluster category). Only when the potential is non-degenerate can these constructions be used to adequately model the corresponding cluster algebra. This lies at the foundation of the additive categorification of (skew-symmetric) cluster algebras. In addition, one can draw significant conclusions on the cluster algebra of a quiver by investigating the Jacobian algebra of a non-degenerate potential on it, as done in [12]. For such applications to cluster algebras, it is important to know that there exists a non-degenerate potential on every quiver.

Indeed, the existence of a non-degenerate potential over an uncountable ground field was established in [11]. However, the proof is not constructive and except for a few classes of quivers, *it is not known how to write a non-degenerate potential on a given quiver*. Among the quivers for which an explicit form of a non-degenerate potential is known we could mention those belonging to the mutation classes of Dynkin quivers [8, 11]; those arising from triangulations of surfaces [13, 20, 21]; those arising from reduced expressions in Coxeter groups [6, 7, 16]; certain McKay quivers [9].

Moreover, it is not known in general *whether a non-degenerate potential on a given quiver is unique*. Here, uniqueness is considered up to *right equivalence*, which is roughly speaking an automorphism on the space of potentials on the quiver. We have already shown [22] the uniqueness of a non-degenerate potential for the quivers belonging to the class $\mathcal{P}$ of Kontsevich and Soibelman [19]. The question of uniqueness has been also settled for many finite mutation classes of quivers [15, 17].

The last finite mutation class for which the answers to the above questions were not known was that of the quiver X_7, discovered in [10]. Its mutation class consists of two quivers and it does not belong to the class $\mathcal{P}$, so no explicit description of a non-degenerate potential was known. Moreover, it has been conjectured [15] that it possess more than one non-degenerate potential.

2 Research Objectives and Expected Significance

A systematic investigation of the above problems for the quiver X_7 was initiated in the Ph.D. thesis of Abeer Shkerat done under my supervision. So far we have been able to define two potentials W and $\widetilde{W}$ on X_7 and prove the following:

- W and $\widetilde{W}$ are non-degenerate.
- The Jacobian algebra $\mathcal{P}(X_7, W)$ is finite-dimensional if and only if the characteristic of the ground field is not equal to 2. In this case, it is symmetric and thus shares many properties with those associated to quivers arising from triangulations of closed oriented surfaces.

  This seems to be the first instance of a non-degenerate potential whose Jacobian algebra behaves differently according to the characteristic of the ground field.
- The Jacobian algebra $\mathcal{P}(X_7, \widetilde{W})$ is always finite-dimensional.
- The mutation of (X_7, W) at the central vertex yields a potential similar in form to those arising from dimer models on closed oriented surfaces, however it is not of such form.

Based on these findings, we propose several directions for further research, which are elaborated in the next section.

1. Study the Jacobian algebras $\mathcal{P}(X_7, W)$ and $\mathcal{P}(X_7, \widetilde{W})$.

   Sample questions to consider are the symmetry of the algebra $\mathcal{P}(X_7, \widetilde{W})$ and whether it is isomorphic to $\mathcal{P}(X_7, W)$ in characteristic different than 2.
2. Find additional non-degenerate potentials on X_7.

   As $\widetilde{W}$ is a certain deformation of the potential W, we aim to consider other deformations of W, investigate their non-degeneracy and extract numerical invariants of their Jacobian algebras with the ultimate goal of classifying all the non-degenerate potentials on X_7.
3. Develop a theory of dimer models on non-orientable surfaces.

   In order to explain the varying behavior of the Jacobian algebras $\mathcal{P}(X_7, W)$ according to the characteristic of the ground field on the one hand, and the similarity of this potential to those

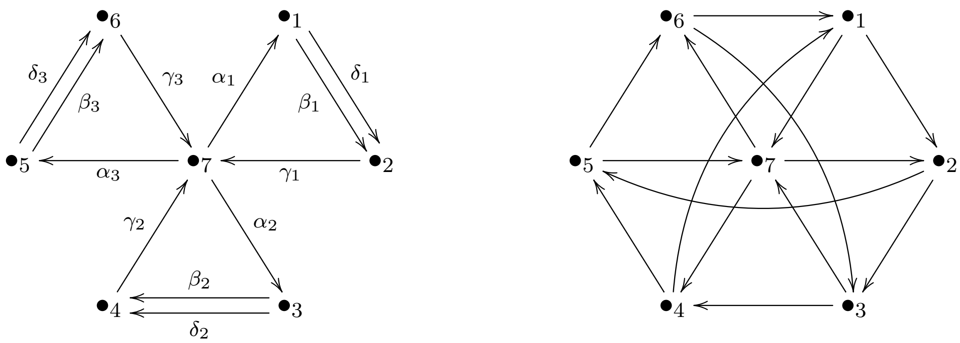

Figure 1: The two quivers X_7 (left) and X'_7 (right) in the mutation class of X_7.

arising from dimer models on the other hand, we propose a geometric model for X_7 by a dimer model on the real projective plane.

To this end we will first develop a general theory of dimer models on closed non-orientable surfaces, that is, a method to construct a quiver with potential from a graph on such surface whose complement is homeomorphic to a disjoint union of discs. Contrary to the classical setup, the surface is no longer orientable and the graph is not necessarily bipartite, which will impose additional conditions on the feasibility of such constructions, that we will explore.

In addition, we propose certain families of graphs on the real projective plane which generalize the graph corresponding to the potential W we constructed and are thus expected to yield quivers with potentials whose Jacobian algebras share similar properties.

3 Detailed Description of Proposed Research

3.1 Preliminary Results

We have been able to explicitly construct a non-degenerate potential on the quiver X_7, without any assumption on the ground field. In fact, the potential we constructed is a sum of cycles, and hence it is even defined “over the integers”. Details will appear in the forthcoming thesis of my Ph.D. student Abeer Shkerat.

The mutation class of the quiver X_7 consists of two quivers, which are shown in Figure 1. The quiver X_7 itself is the left one, and we denote the right one by X'_7. Note that $X'_7 = \mu_7(X_7)$ and moreover $X_7 \simeq \mu_i(X_7)$ and $X'_7 \simeq \mu_i(X'_7)$ for any vertex $1 \leq i < 7$.

A potential on X_7 has been suggested in the physics literature [1], but without any rigorous analysis. Our potential seems to differ from that potential. In order to define it, consider the following six 3-cycles $B_1, B_2, B_3, \Delta_1, \Delta_2, \Delta_3$ on X_7 given by

$$B_i = \alpha_i \beta_i \gamma_i, \qquad \Delta_i = \alpha_i \delta_i \gamma_i, \qquad (1 \leq i \leq 3).$$

Denote the characteristic of a field K by $\operatorname{char} K$.

Theorem. *Let K be a field. Consider the potential*

$$W = B_1 + B_2 + B_3 + \Delta_1\Delta_2 + \Delta_2\Delta_3 + \Delta_3\Delta_1 \tag{3.1}$$

on X_7 and let $\Lambda = \mathcal{P}(X_7, W)$ be its Jacobian algebra. Then:

(a) The potential (X_7, W) is non-degenerate.

(b) If $\operatorname{char} K = 2$, then Λ is infinite-dimensional over K.

(c) If $\operatorname{char} K \neq 2$, then Λ is finite-dimensional and symmetric over K.

Recall that an algebra Λ is *symmetric* if Λ and $\operatorname{Hom}_K(\Lambda, K)$ are isomorphic as bimodules over Λ.

The above result yields an explicit description of a non-degenerate potential on the other quiver X'_7 as well. Namely, the mutation of $\mu_7(X_7, W)$ at the vertex 7 is right equivalent to (X'_7, W') where W' is the potential on X'_7 consisting of the six triangles passing through the vertex 7 and three quadrangles passing outside 7. Since there is at most one arrow between any two vertices in the quiver X'_7, any path of length m may be identified with the sequence of vertices $(i_0, i_1, i_2, \dots, i_m)$ it traverses, so W' can be written as:

$$W' = 7237 + 7257 + 7457 + 7417 + 7617 + 7637 + 12341 + 34563 + 56125. \tag{3.2}$$

Part (a) of the theorem is proved by analyzing all the finitely many possible mutations. One has $\mu_i(X_7, W) \simeq (X_7, W)$ and $\mu_i(X'_7, W') \simeq (X'_7, W')$ for any $1 \leq i < 7$, so the mutation class consists of exactly two quivers with potentials.

By a general result on symmetric Jacobian algebras [23] we deduce the following:

Corollary 1. *If $\operatorname{char} K \neq 2$, then the Jacobian algebra $\Lambda' = \mathcal{P}(X'_7, W')$ is finite-dimensional, symmetric and derived equivalent to Λ.*

Since there is a non-degenerate potential on X_7 whose Jacobian algebra is infinite-dimensional, we can invoke the results of [5] and deduce the following:

Corollary 2. *None of the quivers X_7 and X_7' admits a reddening sequence in the sense of [18].*

We remark that the weaker statement that there is no maximal green sequence on X_7 has been previously proved by combinatorial techniques [24].

The latter corollary has some implications on the cluster category. When $\operatorname{char} K \neq 2$, the finite-dimensionality of its Jacobian algebra allows to associate to the quiver with potential (X_7, W) a generalized cluster category $\mathcal{C} = \mathcal{C}_{(X_7,W)}$ in the sense of [2], which is a 2-Calabi-Yau triangulated category with a cluster-tilting object.

Corollary 3. *The exchange graph of cluster-tilting objects in the cluster category $\mathcal{C}_{(X_7,W)}$ is not connected.*

We note that to the best of our knowledge, this is the first instance of a quiver with non-degenerate potential whose Jacobian algebra behaves differently according to the characteristic of the ground field. Previously, there were known examples of quivers possessing several non-degenerate potentials, some of them with finite-dimensional Jacobian algebras and some with infinite-dimensional ones [15, 22], but this property did not depend on the ground field.

Recently, we have been able to show the next result.

Theorem. *Let K be a field. Consider the potential*

$$\widetilde{W} = B_1 + B_2 + B_3 + \Delta_1\Delta_2 + \Delta_2\Delta_3 + \Delta_3\Delta_1 + \Delta_1\Delta_2\Delta_3 \tag{3.3}$$

on X_7. Then:

(a) The potential $(X_7, \widetilde{W})$ is non-degenerate.

(b) The Jacobian algebra $\widetilde{\Lambda} = \mathcal{P}(X_7, \widetilde{W})$ is finite-dimensional.

As a consequence, we can partially confirm the conjecture of [15].

Corollary 4. *If $\operatorname{char} K = 2$ then the potentials W and $\widetilde{W}$ on X_7 are not right equivalent.*

3.2 The Jacobian Algebras

Many aspects of the algebra Λ, such as the computation of its Cartan matrix and its center, have been considered in the Ph.D. thesis of Abeer Shkerat. The above results motivate a similar study of the finite-dimensional algebra $\widetilde{\Lambda}$. We aim to consider the following questions and problems.

- Is the algebra $\widetilde{\Lambda}$ symmetric?
- Compute the Cartan matrix and the center of $\widetilde{\Lambda}$.
- Are the algebras Λ and $\widetilde{\Lambda}$ isomorphic when $\operatorname{char} K \neq 2$?

A negative answer to the last question would confirm the conjecture of [15] also when $\operatorname{char} K \neq 2$.

We note that while there is a natural grading on the algebra Λ which turned out to be very useful in calculations, this does not seem to be the case for the algebra $\widetilde{\Lambda}$.

3.3 Additional Non-degenerate Potentials

Our preliminary results indicate that the quiver X_7 shares many properties with the quivers arising from triangulations of closed oriented surfaces of positive genus with one puncture, namely the inexistence of reddening sequences, the symmetry of Jacobian algebras and the existence of more than one non-degenerate potential.

By now it is known that on each of these quivers there are actually *infinitely many* non-degenerate potentials whose Jacobian algebras are pairwise not isomorphic [17, 23]. We plan to address the question whether a similar phenomenon occurs also for the quiver X_7. In order to do this, we consider *deformations* of the potential W by adding higher order terms, namely potentials of the following form

$$W_f = B_1 + B_2 + B_3 + \Delta_1\Delta_2 + \Delta_2\Delta_3 + \Delta_3\Delta_1 + f(\Delta_1, \Delta_2, \Delta_3),$$

where $f(x_1, x_2, x_3)$ is a power series in the non-commuting variables x_1, x_2, x_3 such that:

- f is a sum of monomials of degree at least 3;
- consecutive variables in each monomial (or its rotation) must be distinct; for instance, $(x_1x_2)^m$ is allowed, but $x_1^2x_2$ or $x_1x_2x_1$ are not.

For instance, $W = W_0$ and $\widetilde{W}$ corresponds to the monomial $x_1x_2x_3$.

One can consider the following two natural problems:

- (Non-degeneracy) Give conditions on f such that W_f is non-degenerate.
- (Equivalence) Given two power series f and g, when are the two potentials W_f and W_g equivalent? More specifically, when are $W = W_0$ and W_f are equivalent?

Concerning the first problem, we were able to compute the mutations of (X_7, W_f). In particular, $\mu_i(X_7, W_f) \simeq (X_7, W_f)$ for $1 \leq i < 7$ and $\mu_7(X_7, W_f) = (X'_7, W'_f)$ where W'_f can be explicitly described using cycles similar to those appearing in the expression (3.2). Moreover, the mutations $\mu_i(X'_7, W'_f)$ for $1 \leq i < 7$ can also be computed, but the main difficulty is that in general they are not equivalent to W'_f anymore.

Concerning the second problem, we note that in order to show that W_f and W_g are not equivalent, it is enough to prove that some numerical invariants of their Jacobian algebras differ.

3.4 Dimer Models on Non-oriented Surfaces

The potential that we found on X_7 has the special property that the answer to the question whether its Jacobian algebra is finite-dimensional or not depends on the characteristic of the ground field. One would like to know whether this is an isolated example or rather a tip of an iceberg.

Problem. Find more non-degenerate quivers with potentials whose Jacobian algebras behave differently according to the ground field.

We propose to develop a theory of dimer models on non-oriented surfaces that will allow, at least in the case the surface is a projective plane, to construct quivers with potentials whose finite-dimensionality of their Jacobian algebras depends on whether the characteristic of the ground field is 2 or not.

Consider the potential W' on the quiver X'_7 given in (3.2). It is a sum of nine cycles, six of them of length 3 and the other three are of length 4. Moreover, *each arrow appears in exactly two cycles*. This reminds us of quivers with potentials arising from dimer models, we refer to the monograph [4] and the recent survey [3].

Dimer models are built from bipartite graphs on oriented surfaces. In order to construct a graph from the quiver with potential (X'_7, W'), we take its *dual graph*; its *nodes* are the cycles in W' (there are 9 of them) and its *edges* are in one-to-one correspondence with the arrows of X'_7 (so there are 15 of them). If an arrow α appears in the cycles c and c', we connect the corresponding nodes by an edge $c \overset{\alpha}{\text{——}} c'$. The dual graph of (X'_7, W') we obtain in this way is shown in Figure 2.

Apart from the nodes and the edges, there is also a higher dimensional structure consisting of the *faces*, which correspond bijectively to the vertices of the quiver. In our example each face is a quadrangle or an hexagon, in accordance with the valency of the corresponding vertex.

We observe the following points which distinguish the graph from the dimer models considered so far in the literature:

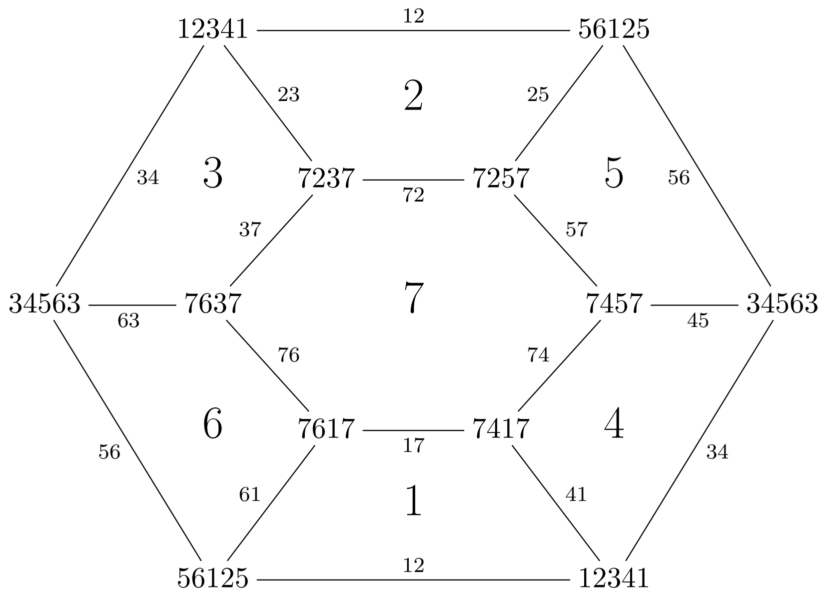

Figure 2: A graph on the projective plane corresponding to the quiver with potential (X'_7, W'). Nodes and edges with the same label have to be identified.

- The graph is not bipartite (for instance, there is a cycle of length 3 passing through the node 12341);
- The surface the graph defines by gluing the faces along the edges is not oriented.

  Indeed, its Euler characteristic is $\chi = 9 - 15 + 7 = 1$, i.e. that of the projective plane $\mathbb{RP}^2$.

One can therefore consider a graph on a non-oriented surface whose complement is homeomorphic to a disjoint union of discs (in order to have the structure of faces) and then try to apply a reverse procedure in order to construct a quiver with potential. It is clear that the vertices of the quiver will correspond bijectively to the faces and its arrows will correspond to the edges, however since there is no partition of the nodes into “black” and “white” and the surface is not necessarily oriented, it is not a-priori clear how to orient each arrow in a consistent manner yielding the required cycles which should correspond to the nodes of the graph.

Problem. Find conditions on arbitrary graphs on surfaces (not necessarily oriented) that will allow to define quivers with potentials in a way which generalizes the known one for bipartite graphs on oriented surfaces.

Let us suggest a family of such graphs on the projective plane. Consider the graph in Figure 2;

we see a central hexagon, surrounded by one layer comprising of six quadrangles. This can be generalized in two directions; firstly, the central hexagon can be replaced by any $(4n+2)$-gon for $n \geq 1$; secondly, instead of just one layer, one can have $m \geq 1$ layers, each consisting of $4n+2$ quadrangles. As in Figure 2, we identify each pair of antipodal edges on the circumference of the outermost layer.

Denote by $v_{n,m}$, $e_{n,m}$, $f_{n,m}$ the numbers of nodes, edges and faces in such a graph. Since half of the nodes (and edges) on the outermost layer are identified with their antipodes, we have

$$\begin{aligned}
f_{n,m} &= 1 + m(4n+2), \\
v_{n,m} &= m(4n+2) + (2n+1) = (2m+1)(2n+1), \\
e_{n,m} &= 2m(4n+2) + (2n+1) = (4m+1)(2n+1),
\end{aligned}$$

so the Euler characteristic equals $v_{n,m} - e_{n,m} + f_{n,m} = 1$.

Denote by $(Q_{n,m}, W_{n,m})$ the corresponding quiver with potential. The quiver $Q_{n,m}$ has $f_{n,m}$ vertices and $e_{n,m}$ arrows, and the potential $W_{n,m}$ is the sum of $v_{n,m}$ cycles, $4n+2$ of them are of length 3 and the rest are of length 4. When $(n,m) = (1,1)$, one gets the quiver with potential (X'_7, W').

We expect the following properties of the Jacobian algebra $\Lambda_{n,m} = \mathcal{P}(Q_{n,m}, W_{n,m})$:

- When $\operatorname{char} K = 2$, the algebra $\Lambda_{n,m}$ is graded and infinite-dimensional;
- When $\operatorname{char} K \neq 2$, the algebra $\Lambda_{n,m}$ is finite-dimensional. We plan to investigate whether it is also symmetric.

Unless $(n,m) = (1,1)$, the mutation class of $Q_{n,m}$ will be infinite and this will pose difficulties in establishing the non-degeneracy of the potential $W_{n,m}$. However, one can restrict to a finite subset of the mutation class obtained by mutating each time only at vertices of valency 4. In the dimer model setup these mutations correspond to *urban renewal* known also as *spider moves* (see [3]). We expect to get quivers with potentials whose Jacobian algebras have the same properties stated above, and plan to investigate the question of derived equivalence of these algebras with the initial algebra $\Lambda_{n,m}$.

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