1. Scientific background

Mathematics is key to many fields and is relevant to the vast majority of tertiary students. Yet most students shy away from this discipline, viewing it as a field that merely deals with quantities. Beyond being a factor in student undergraduate course choice, this bias also limits the ability of students to realize the full potential of the fields in which they have chosen to major. In many faculties, students are not aware of what mathematics has to offer—that it indeed deals with quantities but also with patterns, structures, changes and space.

Perhaps the greatest lack of undergraduate math concepts can be found in the faculties of art and design which usually do not include mathematical ideas in their curriculum (except basic geometry). Designers and especially industrial designers, are educated in the academy to innovate new products and features. They are driven by this objective to push their boundaries with the help of other scientific domains, including materials engineering (especially mockups in 3d printing), artificial intelligence, mechanics, and other fields. From my point of view mathematics has a variety of tools that are just waiting for the right open mind designer to be applied; algebra, topology, etc.. Mathematical theoretical tools may be considered not only by patterns in the finishing process of a given product but also in the initial steps of planning a product. In some of the cases questions as "Is it possible to define a product X with properties Y" can be answered in the planning process by mathematical justifications, as have been described in [3].

In this proposition, we will focus on the symbiotic relationship between mathematics and design, and how the tools of each of these distinct fields can provide scientific innovations to the other.

My vision, In a similar way to mathematical-physics, mathematical-biology, to provoke and establish the mathematical-design field and show how important it is that intermediate math will be part of the curriculum in different studies of art and design.

2. Objectives and significance

The main objective of this project is to connect mathematics, art, and design. We will show how mathematics with computational tools can define innovation in design and art, and more surprisingly, how design concepts can inspire to define new mathematical ideas. In this proposition, the focus is in the following three topics:

(1) **Classifying and defining songs as three-dimensional objects (Aim 1).** Can a given song in Western music be modeled as a collection of curves, surfaces, or even be defined as a tangible object? If they indeed can be modeled in a pure mathematical fashion, can we sort songs by equivalence relation? This research involves music, industrial design, differential geometry, algebra, and topology.

(2) **Gradient topology (Aim 2).** Gradient is an important concept in mathematics, surprisingly this concept is also well defined from a designer's point of view as a soft color-changing (which contains the mathematical definition) in a given image. As a mathematician this simple design-gradient,

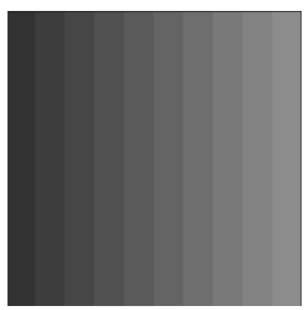


Figure 1. A classical visual gradient in gray color space. In this case can be considered as the fundamental polygon of a cylinder, since the upper side and the lower side are in the same color direction.

as in Fig. 1, although it has a geometrical property, reminds us the construction of a cylinder from the respective fundamental polygon, which is obtained by attaching the identical edges. This observation led us to think is it possible to define design-gradients for different topological surfaces (torus, Klein bottle, etc.), and if the answer is yes, can we give an upper and lower bound to the number of each gradient that exists for each topological surface. This research involves design, topology, combinatorics, and complexity; and all is influenced by design concepts.

(3) **Defining dynamical tiling's in industrial design (Aim 3)**. It turns out that algebraic structures can help designers in the planning steps to know if a dynamical transformation of the components can be obtained, all by a respective mechanism which defines movement between different patterns/arrangements, each of which accomplishes a different goal, as has been introduced in [1, 3]. It can be applied to folding tables (reduction and expansions), lightning systems (exposure and hiding), and more. I intend to generalize this result not only for planar patterns but also for spherical patterns and especially for geodesic domes. This research involves industrial design (three-dimensional visualization and mechanicsm construction), differential geometry, and groups.

3. About the lab

The Lab for Designing Mathematics, which I head, is a multidisciplinary research lab focused on ideas that involve advanced mathematics and design (especially industrial design). My team uses diverse tools from various faculties to achieve this aim, from computer science to industrial design. To advance our goal, we collaborate with various departments on campus, such as design (first and second degree), computer science (first and second degree), applied mathematics; and other institutes. The team aims to connect research fields that are traditionally perceived as starkly different, e.g., math, design including music and art. We are driven by the belief that our efforts can aid in the dissemination of intermediate mathematics concepts among designers and artists and, of course, help apply non-trivial mathematical ideas not only in traditional connections as physics or computer-science; but also in design fields. Lastly, I believe it may even instill in the average design/art student an appreciation, or even a passion, for the field of mathematics.

4. Detailed description of the research

4.1. **Aim 1: Classifying and defining songs as three-dimensional objects.**

In [5, 8], a framework for mapping a chorus onto a three-dimensional structure by transforming guitar choruses of Beatles songs into respective curve (with constraints). It focused on exploring the total curvature of the chorus curve, which can define similarity between different choruses. It also can help the performer to determine the geometric representation they aim to convey through the number of loops and the direction of the curve. In addition, viewing the curve, as Fig. 2, offers non-professional audiences a glimpse into the complexity of composing.

In this project, I intend to produce and formulate the following concepts

- Respective to the curves obtained in [8]. The oriented polygonal curve is obtained by a sequence of vertices. For each two adjacent vertices, an harmonical distance will be defined. By the help of an industrial designer, we will define the physical curve with changing materials along the vertices which will represent best the harmonical distance. With this approach, we hope we can not only hear the song but also feel its respective harmonic. This idea can be especially important those who suppfering from hearing loss.
- As have been done in [8] for the chorus, we would like to generalize this idea for the whole song, i.e., defining a curve for the chorus, verse, etc. In this case, we may get knot alike curve structure which we believe can be explored. Lastly, with the help of an industrial designer we will produce this physical object.
- Generalize the idea of curves to surfaces, i.e., each song will be approximated as a surface. From a topological point of view, each of these surfaces is determined by properties as Euler characteristic number, orientability, etc. and can be produced as a physical object.

4.1.1. *Rationale.* This project strives with the help of mathematics and industrial design to transform music into a tangible physical object. Understanding the structure of music typically requires a great deal of study. In this work, with the help of design tools, we will convert music into objects by relying on their respective chords, which reflect the complexity in a given song. This 3D visualization can offer non-musicians a glance into how complicated or simple a piece of music is. We will explore famous songs, especially in Western music, where the song is generally comprised of a verse and chorus. We will show that some songs that sound utterly different can, in fact, be represented by the same object.

4.1.2. *Work plan.* Music can be written as triads (a, b, c). The set of all 24 major and minor triads can be thought of as an abelian group isomorphic to the group $\mathbb{Z}_{12} \times \mathbb{Z}_2$. [2, 9] gives a mathematical formulation for triads.

This research will show that "songs" can be simulated as a collection of curves or surfaces. In this study, the initial input will be songs composed by a sequence of triads (without voicing).

A song in Western music is defined (generally) as a chorus and verse, each of which defines a sequence of triads. Each chorus or verse will be considered a closed curve (by defining the location of each triad, which can define a closed curve with a respective total curvature) or a surface with respective topological properties, such as a Gauss curvature and geodesic curvature (similarly to what was done in [4]).

From the curves point of view an approximation to the collection of curves will be given. In addition, this sequence of points can be approximated as a surface, each can be sorted topologically by properties as Euler characteristic number, orientability, and boundary, for more details about this classification see [11].

The result will be the sorting of songs by the equivalence relation of curves or surfaces; all representations will be visualized with the help of industrial designers to represnt best the harmony by a respective material between adjacent vertices. The team will have to exhibit these ideas as an object and to portray to diverse audiences the complexity or simplicity of music.

4.1.3. *Preliminary results.* In general given a triad t_i where $1 \leq i \leq n$, i.e., the chorus has n triads. It can be written as the sequence $t_1, \ldots t_n$, i.e., $t_1 \to t_2, \ldots, t_{n-1} \to t_n$, which defines a polygonal curve. In [8], this chorus polygonal curve has been explored by the total curvature point of view.

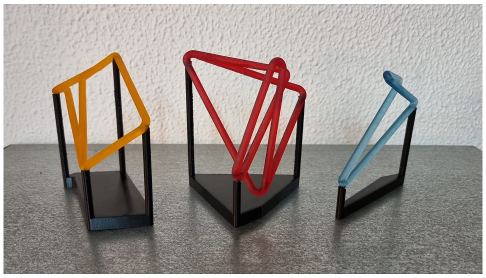


FIGURE 2. Our representation of three Beatles songs which we transformed into $3D$ physical objects, from left to right: Hello goodbye, All you need is love, Like dreamers do. $3D$ printing Pla/Sla.

In this proposition, for each chorus which is defined by a curve, we will define for each two adjacent chords/vertices a harmonic distance, as has been described in [16]. With the help of an industrial designer, each harmonic distance will be represented by a respective material, if the harmonic distance is small, the edge will be represented by a soft material and pleasant color, if the harmonic distance is 'large' then the edge will be represented by a rough or spiky material. The result is an object with diverse materials, where touching along the object (along the curve) will lead to the chorus feel. We need to decide what are the right materials and metric.

4.1.4. *Pitfalls.* Due to the results in [8], the aim of defining the feeling of the chorus along the curve with the help of an industrial designer is reasonable.

Exploring songs as a collection of curves is challenging. In this case, the main question is what is the importance of the knot? Will it give any real insight to musicians or better understanding to non-professional audience?

In the case of exploring chorus as a surface the resuly will be obtained by the right triangulation. The result in this case can be in contradiction to the results in [8].

4.1.5. *Expected outcomes and impact.* This research should yield a new method for generating a tangible visualization (curves or surfaces) of songs. It will present to a wide audience why music can be considerably complex and give an idea of how music relates to mathematical ideas such as algebra, geometry and topology. It can be transformed from a hearing experience to one of touching an object that reflects the music's internal harmony. It can also provide musicians with a tool for portraying the diverse nature of their music and offer non-musicians but yet experienced audiences a glance into the complexity of music. Last, these kinds of tools, with our technique and the right materials, may one day allow deaf people to enjoy a song, not by hearing it but by feeling it.

4.2. **Aim 2: Gradient topology.** In art and design gradient is a smooth transition from one color to another. It gives the artist/designer to add soft feel and uniqueness to their object. It also, has eye-catching and memorable visual designs, while solid colors can be thought of as stiff colors. It can be applied in cases where the artist trying to transmit shade or light on a given product, create a focal point, etc., for more details about surface classification see [10, 13].

It led us to think (mathematicians and designers), can we formulate different gradients for color by a mathematical rule? Even though usually gradient is related to geometry our approach leaned on fundamental polygons in topology, which represent different surfaces classification (torus, Klein bottle, etc.). Many works have been done about Suduku which can be related to visual gradient solution for a given matrix. We believe that many ideas and solutions can help us and vice versa, see [7, 12, 17]. We believe that this research shows the importance of involving other fields that mathematicians are not familiar with, in this case design, which inspires the formulating of new mathematical explorations.

4.2.1. *Rationale.* We define for the first time a language of visual gradients which is influenced by design ideas combined with topology and combinatorics. We will show how to construct different types of visual gradients given a fundamental polygon for different initial states, as Möbius, Klein bottle, etc.. Further, given an initial state of a gradient (which will be defined in Section 4.2.3), we will show which topological gradients constructions can be obtained.

Once we defined good language we will be able to define harder questions and deeper results and our hope is that our point of view will open the door for systematic research in this area, both from mathematics and design.

Lastly, topology has a lot to offer to art and design, as in [14, 15], we will show how it can be a real tool to the average designer.

4.2.2. *Work plan.* We first need to define a language that connects between visual gradients and a fundamental polygon. We also need to define a good filteration for the space of possible initial states which is identified with a partial defined matrix. In the next step, we will try to classify which topological visual gradients could be obtained from each step in the filtration. Lastly, the industrial designer in the team will apply our formulation for a given product-which is homeomorphic to a given surface.

4.2.3. *Preliminary results.* These are the basic definitions and initial resuts.

Definition 4.1. Given a cell (i, j) in a matrix $n \times n$. We define cell's neighbors as all the cells it borders with horizontally, vertically, or diagonally $pixel(i, j)$, $neighbor(pixel(i, j))$.

Notice that by this definition an interior pixel have eight neighbors, see Fig 3. In gray scale continuous color scale will be defined starts at zero and increasing

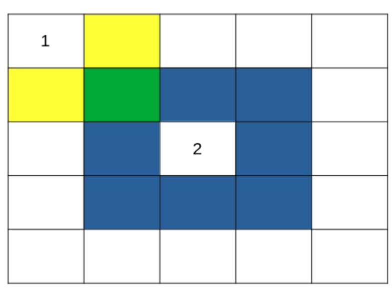


FIGURE 3. Cell 1 neigboors in yellow, cell 2 neigboors in blue, green is a common neigboor to both.

is a constanst natural number C, such that $C \cdot n \leq 255$. Examples $0, 1, 2, \ldots 255$, $0, 5, 10 \ldots 255$.

Definition 4.2 (Visual Gradient)**.** Given a pixel (i, j) and a continuous color scale. If for each pixel neighbor

$$\|pixel(i,j) - neighbor(pixel(i,j))\| = 0 \text{ or } C$$

To explore this connection we define the following cases

Definition 4.3 (Initial term for gradients topology)**.** An initial state gradient is a partially field matrix.

Definition 4.4. A matrix $n \times n$ is called a full initial state if all the borders matrix are full. Partial initiate state if it only partial given.

For example see Fig. 5.

1	2	3	2	3
2				2
3				1
2				2
3	2	1	2	1

(A) Full initial state. In this case it may lead to a Klein buttle

1	2	3	2	3
3	2	1	2	1

(B) Partial initial state. It may lead to different topological surfaces.

FIGURE 4. This initial states of gradient can define the topological surface.

Definition 4.5. An initial stat will be called monotone if the initial values in each row or columns defines strictly monotonic sequence.

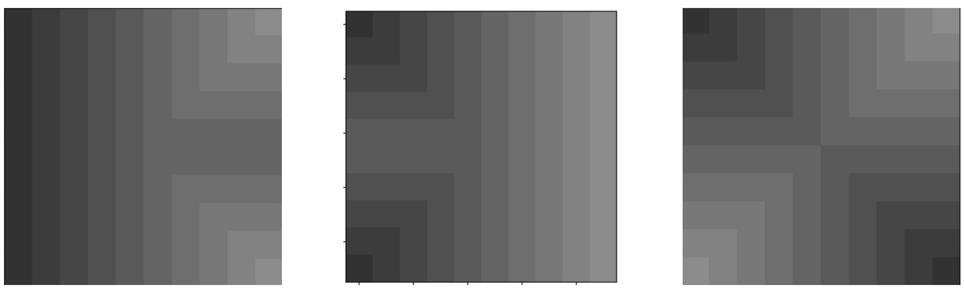
Definition 4.6. Let X be a topological surface. An initial state A will be called an initial state of X if the edges of A defines the fundamental rectangle of X.

The following definition will connect the visual gradient and topology.

Definition 4.7 (An initial topology gradient)**.** Given a topology surface X with a respective fundamental polygon gradient. An initial X state for a gradient matrix n, is an initial state aligned with the fundamental polygon.

Now we are ready to define visual gradients as respective topological surfaces. We will start with the most intuitive one, a cylinder.

Definition 4.8. A cylinder gradient is a gradient which is defined by the fundamental polygon with only two parallel edges in the same direction, i.e., represent a cylinder $\forall 1 \leq j \leq n : pixel(1, j) = pixel(n, j)$. A rotation of this gradient is from the same type.



(A) Cylinder (B) Cylinder (C) sphere

FIGURE 5. Examples of visual gradients which the initial state defines a topological surface.

In Figures 5a and 5b, we can see that the cylinder gradient is not unique, and made us wonder 'Can we give an upper and lower bound to the number of gradients that can be obtained for a given topological surface?'

We will give a glance to the powerful of the language we trying to formulate.

Proposition 4.9. *Given a monotonic initial state for a given image $n \times n$ and $n \leq 256$, then X must be a sphere.*

We decide to omit the proof, an example for sphere gradient see Fig. 5c.

4.2.4. *Pitfalls.* This research involves different disciplines, each of which requires a specialty. We may add to the team more researchers from mathematics, computer science and design to make sure that progress is obtained.

4.2.5. *Expected outcomes and impact.* The expected outcome is a systematic math treatment for the visual gradient. This proposal has many different research outcomes. From design point of view, understanding the mathematics can help the designer well customize which visual gradient is best to apply in a given product. From the mathematical point of view, our formulation connects fields which supposed to be so different from one to another: geometry, topology, and combinatorics. In addition, there are hundreds of publications exploring Suduko, we believe that the scientific community in this field will find a lot of interest in our language. Equally important, it will encourage mathematician to open their mind and learn diverse fields (in our case design/art) that might not seem related to their main research. It may also be considered in the curriculum for undergraduate students in mathematics or design, trying to show how intermediate mathematics can be applied with the right guidelines.

4.3. **Aim 3: Dynamical Tilling.** Usually, industrial design innovation is related to materials and AI. When designers deal with mathematics it usually relates to the geometric properties of the product, while mathematics has a variety of tools; algebra, topology, etc. This research shows that innovation can come from unexpected places, such as group theory which can define a dynamical tiling between differen stages. This topic is essential to designers to increase their toolbox not only in the finish steps of defining a pattern but also in the initial steps of planning a product which have tilings properties.

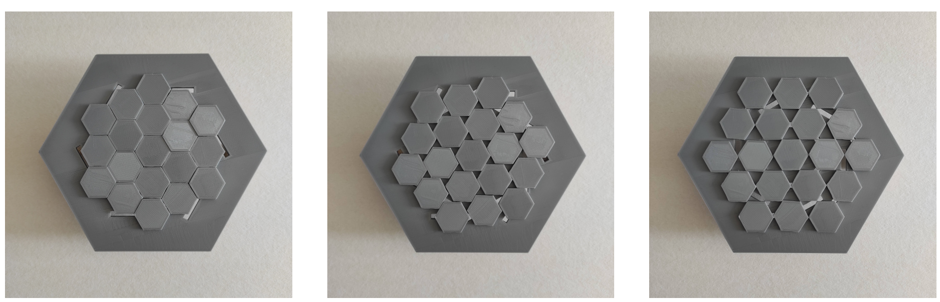
We intend to define a geodesic dome which is obtained by a tiling-group, which can define a dynamical movement-based on the respective sub-group. We may consider defining a proper mechanism that determines the movement, as has been done in [3]. As mathematics, our dynamical tiling is not design for a specific task, it is a concept that can be applied to the specific demand of a designer who intend to innovate a product with certain properties, such as reduction and expansion, exposure or hiding, and more.

4.3.1. *Work plan.* In [6], the signatures of patterns/tilings and the respective groups relationship have been defined. In this project, we intend to apply some of these relations and define a dynamical movement between different stages in the plane, but especially for geodesic domes. The mathematical exploration is based on the relations of spherical patterns (which are related to domes), i.e., the respective groups and sub-groups. To obtain the geodesic dome we first consider a different deform plane model, as in Fig. 7, which we believe can give us the knowledge of how to define the respective mechanism.

4.3.2. *Preliminary results.* In [3] we discussed the relation in planar patterns, and show how a dynamical movement can be obtained between the hexagonal regular lattice which is defined by reflections, rotation, and translation (signature $*632$), and its sub-group which is defined by rotation and translations (signature 632), as demonstrated in [1].

Before moving to spherical patterns, we first delve into planar patterns, since it can be considered as a local approximation of the spherical pattern. We need to decide which planar pattern and relation is best to deformed, not only by mathematics but also in the material design.

In Fig. 7, given a local representation of a spherical pattern with the respective signature, by this basic model, we trying to evaluate what is the 'optimal' movement with the help of the respective groups.



(A) Initial state, signature $*632$

(B) Middle state leads to signature 632

(C) The model open stage, signature $*632$

FIGURE 6. Our planar model defines with a proper mechanism a dynamical movement between tilings (made from PLA). When it reaches the end of the rail it gives an extended pattern of signature $*632$.

In Fig 8, planar patterns have been considered by various materials. In Fig. 8a, we consider paper and brass fasteners which is flexible enough, such that a deformation can be obtained. We hope that it will give us another direction to spherical pattern approximation. In Fig. 8b we use laser-cut wood and brass fasteners; which leads to a scissor linkage mechanism for a pentagon and that opens and closes. Since, in some of the spherical patterns pentagons play a major role, We believe that this kind of experiments can lead to the desired mechanism. In Fig. 8d, combining the previous steps, we try to approximate signature $*532$.

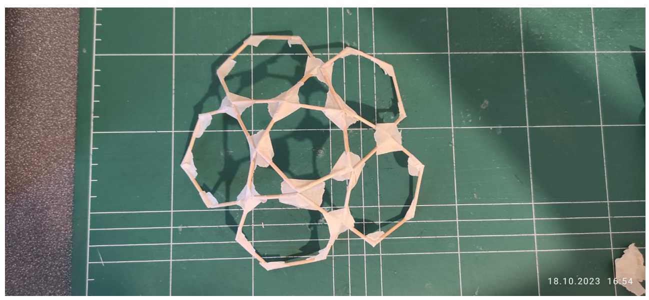
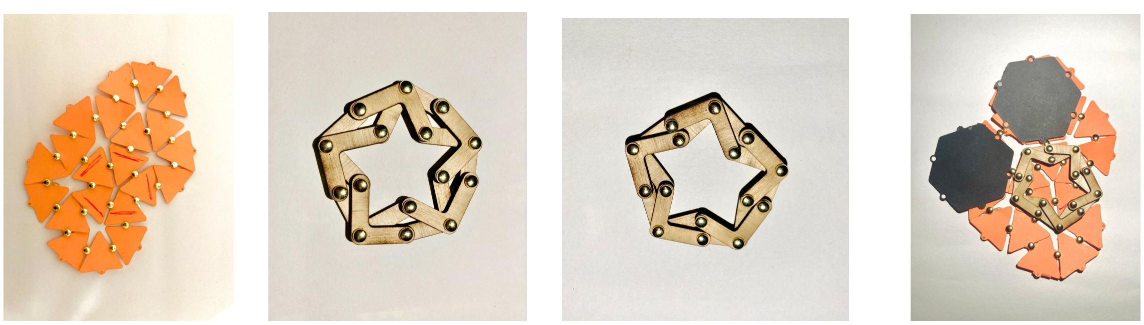


FIGURE 7. Our first naive experiment in spherical patterns, signature $*532$. By playing the object, as dismissing the triangles, we trying to predict the mechanism.



(A) (B) (C) (D)

FIGURE 8. Exploring movement, deformation and mechanism.

4.3.3. *Pitfalls.* If we would like the dynamical geodesic dome to be applied to real uses the construction needs to be efficient. As an example, if the mechanism is inside the dome, the inner side of the dome will be lack of space.

4.3.4. *Expected outcomes and impact.* This research gives new mathematical tools to the average industrial designer. The designer chooses a given pattern in which a movement is required in a physical object to a given purpose. By our mathematical language, the pattern can be transformed to the respective fundamental area, tilying and signature. Each of which (pattern), has sub-group that will be the natural candidate for the dynamical movement which is required.

5. Summary

I seek to find a symbiotic relationship between design and math in this project. In Section 4.1 industrial design by mathematical foundations helps to give a tangible feel and visual to music. Section 4.2 shows how design definition can help to formulate new ideas in mathematics. Section 4.3 shows how abstract algebriac ideas can be adapted to industrial design.

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