A General Framework for Endowment Effects in Combinatorial Markets

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Abstract

"Losses loom larger than gains" — Daniel Kahneman; Amos Tversky

The *endowment effect*, coined by Nobel Laureate Richard Thaler, posits that people tend to inflate the value of items they own. This bias was studied mainly using an experimental methodology. Recently, Babaioff *et al.* proposed a specific formulation of the endowment effect in combinatorial settings, and showed that equilibrium existence with respect to the endowed valuations extends from gross substitutes to submodular valuations, but provably fails to extend to XOS valuations.

We show that this negative result is an artifact of their specific formulation. To this end, we introduce a principle-based framework that captures a wide range of different formulations of the endowment effect, including the formulation proposed by Babaioff *et al.*. We equip our framework with a partial order over the different effects, which (partially) ranks them from weak to strong. We provide algorithms for computing endowment equilibria with high welfare for sufficiently strong endowment effects, as well as non-existence results for weaker ones. Our main results are the following:

- For markets with XOS valuations, we provide an algorithm so that for sufficiently strong endowment effects outputs an endowment equilibrium with at least half of the optimal social welfare.
- For markets with arbitrary valuations, we show that bundling leads to a sweeping positive result. In particular, if items can be prepacked into indivisible bundles, we provide a polynomial algorithm that, given an arbitrary allocation S, computes an endowment equilibrium with the same welfare guarantee as in S. This can be viewed as a black-box reduction from the computation of an approximately-optimal endowment equilibrium to the algorithmic problem of welfare approximation.

1 Introduction

Consider the following combinatorial market problem: A seller wishes to sell a set M of m items to n consumers. Each consumer i has a valuation function $v_i : 2^M \to \mathbb{R}^+$ that assigns a non-negative value $v_i(X)$ to every subset of items $X \subseteq M$. The valuation functions can exhibit various combinations of substitutability and complementarity over items; and as standard, valuations

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are assumed to be monotone ($v_i(Z) \leq v_i(X)$ for any $Z \subseteq X$) and normalized ($v_i(\emptyset) = 0$). Each consumer i has a *quasi-linear* utility function, meaning that her utility for a bundle $X \subseteq M$ that she pays $p(X)$ for is $u_i(X, p) = v_i(X) - p(X)$. An allocation is a vector $S = (S_1, \ldots, S_n)$ of disjoint bundles of items, where S_i is the bundle allocated to consumer i. The social welfare of an allocation S is the sum of consumers' values for their bundles, i.e., $SW(S) = \sum_{i \in [n]} v_i(S_i)$. An allocation that maximizes the social welfare is said to be socially efficient.

A classic market design problem is setting prices so that socially efficient outcomes arise in "equilibrium". Arguably, the most appealing equilibrium notion is that of a Walrasian Equilibrium (WE) [Walras, 1874]. A WE is a pair of allocation $S = (S_1, \ldots S_n)$ and item prices $p = (p_1, \ldots, p_m)$, where each consumer maximizes her utility, i.e.,

$$v_i(S_i) - p(S_i) \geq v_i(T) - p(T)$$

for all $T \subseteq [m]$, and the market clears, namely all items are allocated.[1] A WE is a desired outcome, as it is a simple and transparent pricing that clears the market. Moreover, by the "First Welfare Theorem", every allocation that is part of a WE maximizes the social welfare[2].

Unfortunately, Walrasian equilibria exist only rarely. In particular, they are guaranteed to exist for the class of "gross substitutes" valuations [Kelso Jr and Crawford, 1982], which is a strict subclass of submodular valuations; and in some formal sense, it is a maximal class for which a WE is guaranteed to exist [Gul and Stacchetti, 1999]. Given the appealing properties of a WE, it is not surprising that various approaches and relaxations have been considered in the literature in an attempt to ameliorate the non-existence problem.

The endowment effect. The *endowment effect*, coined by Thaler [1980], posits that consumers tend to inflate the value of the items they own. This phenomenon was later validated by experiments, which realized and quantified the magnitude of the effect [Knetsch, 1989; Kahneman et al., 1990; List, 2011, 2003]. By now, it is widely accepted that the endowment effect is apparent in many markets.

Yet, as far as we are aware, the endowment effect has been studied mainly via experiments. Recently, Babaioff, Dobzinski and Oren [2018] (henceforth, Babaioff *et al.*) proposed a formal model for studying the endowment effect. In their work they take a behavioral economic perspective, and harness the endowment effect in order to extend market stability and efficiency. In this work, we introduce a new framework that provides a more flexible formulation of the endowment effect, which allows us to generalize and extend their work to richer settings.

Babaioff *et al.*'s formulation. Babaioff *et al.* propose capturing the endowment effect in combinatorial settings by formulating an *endowed valuation* function. Given some valuation function v, and an endowed set $X \subseteq M$, the endowed valuation function, parameterized by α, assigns the following real value to every set $Y \subseteq M$, referred to as the endowed valuation of Y with respect to X:

$$v^X(Y) = \alpha \cdot v(X \cap Y) + v(Y \setminus X \mid X \cap Y), \tag{1}$$

where $\alpha \geq 1$ is the *endowment effect parameter*, and $v(S \mid T) = v(S \cup T) - v(T)$ denotes the marginal contribution of S given T for any two sets S, T. The idea behind this formulation is that the value of items already owned by the agent ($X \cap Y$) is multiplied by some factor α, while the marginal value of the other items ($Y \setminus X$) remains intact.

[1] More precisely, unallocated items have price 0.

[2] Moreover, every allocation that is part of a WE maximizes welfare also over all feasible *fractional* allocations [Nisan and Segal, 2006].

An *endowment equilibrium* is then a Walrasian equilibrium with respect to the endowed valuations, i.e., a pair of allocation $S = (S_1, \dots S_n)$ and item prices $p = (p_1, \dots, p_m)$, where each consumer maximizes her endowed utility:

$$v_i^{S_i}(S_i) - p(S_i) \geq v_i^{S_i}(T) - p(T),$$

and the market clears.

The main result of Babaioff *et al.* is that when consumers' valuations are submodular and $\alpha \geq 2$, there exists an endowment equilibrium that gives a 2-approximation to the optimal (even fractional) social welfare with respect to the original valuations. They also show that the existence result does not extend to the more general class of XOS valuations. In particular, for every $\alpha > 1$, there exists an instance with XOS valuations that does not admit an endowment equilibrium.

The specific function given in Equation (1) is one way to formulate the endowment effect in combinatorial settings, but certainly not the only one. For example, suppose a consumer is endowed some set X, a-priori, it is not clear how to reevaluate some set $Z \subset X$, subject to the endowment effect. Babaioff *et al.* established non-existence result for the case where $v(Z)$ is multiplied by some parameter α. Can a more flexible formulation of the endowment effect circumvent this impossibility result?

1.1 A New Framework for the Endowment Effect

In this section we provide a new framework for various formulations of the endowment effect; our framework is based on fundamental behavioral economic principles. Beyond circumventing impossibility results, our framework seems the right way to treat this problem, as there is no single formulation that fits all scenarios. Specifically, our framework allows reasoning about different ways of defining the value of a subset Z of an endowed set. We hope that our work will inspire further discussion regarding meaningful endowment effects in combinatorial settings, as well as experimental work that will shed more light on appropriate instantiations for different scenarios.

A crucial component of our framework is a partial order $\prec$ over endowment effects, which is *stability preserving*; i.e., given two endowment effects, $\mathcal{E}, \mathcal{E}'$, such that $\mathcal{E} \prec \mathcal{E}'$, a Walrasian equilibrium with respect to the endowed valuations according to $\mathcal{E}$ is also a Walrasian equilibrium with respect to the endowed valuations according to $\mathcal{E}'$ (Corollary 3.7).

As in previous work, we take a "two step" modeling approach, i.e., a consumer has a valuation function v prior to being endowed a set X, and an *endowed valuation* function v^X after being endowed a set X, which describes the inflation in value due to the endowment effect.

Our framework is based on two basic principles, described below.

The "loss aversion" principle. The loss aversion hypothesis is presented as part of prospect theory and is argued to be the source of the endowment effect [Kahneman et al., 1990, 1991; Tversky and Kahneman, 1979]. This hypothesis claims that

People tend to prefer avoiding losses to acquiring equivalent gains.

The loss aversion principle can be formulated as follows:

$$v^{X \cup Y}(X \cup Y) - v^{X \cup Y}(Y) \geq v^Y(X \cup Y) - v^Y(Y) \qquad \forall X, Y \subseteq M \tag{2}$$

The left hand side term signifies the loss incurred due to losing a previously-endowed set X, while the right hand side term signifies the benefit derived from being awarded a set X not having been owned before. The loss aversion inequality states that the loss incurred due to losing X is greater than the benefit derived from getting X.

The "seperability" principle. The additional principle, proposed by Babaioff *et al.*, states that the endowment effect with respect to set X should maintain the marginal contribution of items outside of X intact. That is, given set $Y \subseteq M$, only the value of items in $X \cap Y$ may be subject to the endowment effect. This principle is formulated as follows:

$$v^X(Y \setminus X \mid X \cap Y) = v(Y \setminus X \mid X \cap Y) \qquad \forall Y \subseteq M \tag{3}$$

In section 3.1 we show that these two principles imply that the value of set Y for a consumer that is endowed a set X is given by:

$$v^X(Y) = v(Y) + g^X(X \cap Y) \qquad \forall Y \subseteq M,$$

for some function $g^X : 2^X \to \mathbb{R}$ such that $g^X(Z) \leq g^X(X)$ for all $Z \subseteq X$. The function g^X is referred to as the *gain function* with respect to X. It describes the added effect an endowed set X has on the consumer's valuation.

An endowment effect formulation, or in short: an endowment effect, is then given by a collection of functions $\{g^X\}_{X \subseteq [m]}$ that satisfy the above condition. An endowment *environment* is given by a vector of endowment effects for the consumers $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$, where $\mathcal{E}_i$ is the endowment effect of consumer i.

We discuss each effect $\{g^X\}_X$ through the term $g^X(Z \mid X \setminus Z)$ — the additional loss incurred upon losing a subset Z of an endowed set X due to the endowment effect. In Definition 3.6 we provide a partial order over all endowment effects, based on this loss.

The *Identity* and *Absolute Loss* endowment effects. Let us consider the formulation of Babaioff *et al.* within our framework. The endowed valuation with respect to X is

$$v^X(Y) = \alpha \cdot v(X \cap Y) + v(Y \setminus X \mid X \cap Y) = (\alpha - 1)v(X \cap Y) + v(Y).$$

For the case of $\alpha = 2$ (which is the case that drives their positive results), this endowment effect can be formulated in our framework by setting

$$v^X(Y) = v(X \cap Y) + v(Y).$$

In this case, the gain function is defined by $g^X(X \cap Y) = v(X \cap Y)$. Thus, we refer to this endowment effect as the *Identity* endowment effect, and denote it by $\mathcal{E}^I = \{g_I^X\}_X$ where $g_I^X = v$. Note that the additional incurred loss is $g_I^X(Z \mid X \setminus Z) = v(Z \mid X \setminus Z)$.

We are now ready to introduce a different endowment effect, that we refer to as the *absolute loss* endowment effect. In this effect, the gain function with respect to an endowed set X is

$$g_{AL}^X(Z) = v(X) - v(X \setminus Z).$$

I.e., $\mathcal{E}^{AL} = \{g_{AL}^X\}_X$. For this effect, it holds that the additional incurred loss is $g_{AL}^X(Z \mid X \setminus Z) = v(Z)$. For subadditive consumers, this effect demonstrates a "stronger" loss aversion bias than Identity with respect to the relation $\prec$, defined in Definition 3.6.

Consider adding the definition of $\prec$.

Proposition 1.1. *For a consumer with a subadditive valuation v, it holds that $\mathcal{E}^I \prec \mathcal{E}^{AL}$.*

Intuitively, one can imagine that in the absolute loss effect, a consumer amplifies the loss of a subset Z of an endowed set X by "forgetting" the fact that $X \setminus Z$ remains at the consumer's hand.

1.2 Existence of Equilibria and Welfare Approximation

In this section we present our existence and approximation results. Our approximation results hold with respect to the optimal welfare according to the *original* valuations, and even with respect to the optimal fractional allocation.[3]

Recall that Babaioff *et al.* prove that for the Identity endowment effect, every market with submodular consumers admits an $\mathcal{E}^I$-endowment equilibrium that gives a 2-approximation welfare guarantee.

For the larger class of XOS consumers, Babaioff *et al.* show that an endowment equilibrium may not exist even with respect to an endowment effect $\alpha \cdot \mathcal{E}^I = \{\alpha \cdot g : g \in \mathcal{E}^I\}$ for an arbitrarily large α. This negative result may lead one to conclude that while the endowment effect improves stability for submodular valuations, XOS markets may remain unstable even with respect to endowed valuations. However, we show that this negative result is an artifact of the specific formulation chosen by the authors. As established in the following theorem, the stronger Absolute Loss endowment effect leads to existence and approximation results for markets with XOS valuations.[4]

Theorem 1. [Theorem. 4.1] There exists an algorithm such that for every market with XOS consumers, and every initial allocation $S' = (S'_1, \ldots, S'_n)$ returns an $\mathcal{E}^{AL}$-endowment equilibrium (S, p), such that $SW(S) \geq SW(S')$.

The algorithm is a variant of the algorithm used by Fu et al. [2012]; Christodoulou et al. [2016]. A direct corollary of Theorem 1 is that for every market with XOS consumers, every optimal allocation S can be paired with item prices p so that (S, p) is an $\mathcal{E}^{AL}$-endowment equilibrium. Moreover, we show that every $\mathcal{E}^{AL}$-endowment equilibrium guarantees 1/2 of the optimal welfare:

Theorem 2. [Theorem. 3.5]: Every $\mathcal{E}^{AL}$-endowment equilibrium gives at least 1/2 of the optimal welfare.

The theorem above shows that a stronger endowment effect enables extending the equilibrium existence (and approximation) result from submodular valuations to XOS valuations. Can this result be extended further?

One answer, though unsatisfactory, is yes! For example, consider an endowment effect that inflates the value of a set linearly with its size; e.g., $\mathcal{E}^{PROP} = \{g^X(Z) = |Z| \cdot v(X) : X \subseteq M\}$. We show in Section 6 that this effect leads to a sweeping equilibrium existence guarantee for arbitrary valuations. Moreover, every optimal allocation can be paired with item prices to form an $\mathcal{E}^{PROP}$-endowment equilibrium (Proposition 6.1). While this sounds as a strong result, this effect inflates the value linearly in the set's size, which may be as large as $\Omega(m)$. We believe that such inflation is unreasonably high, and misses the whole point of the endowment effect.

Can we get a general positive result with a "reasonable" inflation? In Section 6 we show that for any endowment effect with inflation up to $O(\sqrt{m})$, an endowment equilibrium may not exist for (the strictly-larger-than XOS valuations) subadditive valuations (Proposition 6.2).

1.3 The Power of Bundling

We next study the power of bundling in settings with endowed valuations. A bundling $B = \{B_1, \ldots, B_k\}$ is a partition of the set of items M into k disjoint bundles. A *competitive bundling equilibrium* (CBE) [Dobzinski et al., 2015] is a bundling B and a Walrasian equilibrium in the

[3] Note that, by the First Welfare Theorem, an endowment equilibrium always gives the optimal welfare with respect to the endowed valuations.

[4] Note that "stronger" here is not in the sense of an increased value of α. Indeed, no finite α suffices for such result.

market induced by B (i.e., the market where $B_1, \ldots, B_k$ are the indivisible items). It is easy to see that a CBE always exists (say, bundle all items together, and assign the grand bundle to the highest value consumer for a price of the second highest value). However, while the WE notion enjoys the first welfare theorem, guaranteeing that every allocation supported in a WE gives optimal welfare, no such welfare guarantee applies with respect to CBE [Feldman and Lucier, 2014; Feldman et al., 2016; Dobzinski et al., 2015].

In this paper we introduce the notion of $\mathcal{E}$-endowment CBE, which is a CBE with respect to the endowed valuations, and provide algorithms for computing $\mathcal{E}$-endowment CBEs with good welfare, for any endowment effect $\mathcal{E}$ satisfying a mild assumption.

Equilibrium computation. Babaioff *et al.* showed computational barriers towards computing $\mathcal{E}^I$-endowment equilibria, and raised the following question (recall that $\alpha \cdot \mathcal{E}^I$ denotes the endowment effect that multiplies each gain function $g \in \mathcal{E}^I$ by α):

Are there allocations that can be both efficiently computed and paired with item prices that form an $\alpha \cdot \mathcal{E}^I$-endowment equilibrium for a small value of α?

The analogous question with respect to CBE and a particular endowment effect $\mathcal{E}$ would be: are there allocations that can be both efficiently computed and paired with bundle prices that form an $\mathcal{E}$-endowment CBE. It doesn't take long to conclude that this problem is trivial for any endowment effect with non-negative gain functions (simply, allocated all items to the consumer with highest valuation for the grand bundle). The interesting problem here would be to compute a nearly-efficient CBE, rather than just any CBE[5], and can be formulated as follows:

Are there approximately optimal allocations that can be both efficiently computed and paired with bundle prices, that form an $\mathcal{E}$-endowment CBE for some natural endowment effect $\mathcal{E}$?

Note that for $\alpha \cdot \mathcal{E}^I$-endowment equilibrium, the two problems coincide, as any $\alpha \cdot \mathcal{E}^I$-endowment equilibrium gives α approximation to the optimal social welfare.

We provide the following positive results, which essentially provide a black-box reduction from the problem of computing approximately optimal endowment CBE for significant endowment effects to the classical algorithmic problem of welfare approximation. This result applies to every *significant* endowment effect — where the gain functions satisfy $g^X(X) \geq v(X)$ for all $X \subseteq M$. For example, one can easily verify that $\mathcal{E}^I$ and $\mathcal{E}^{AL}$ are significant with respect to all consumer valuations.

Theorem [Black-box reduction for endowment-CBE]

1. [Thm. 7.4] *There exists a polynomial algorithm such that for* <u>*submodular*</u> *valuations, and every significant endowment effect $\mathcal{E}$ and initial allocation $S' = (S'_1, \ldots, S'_n)$, computes an $\mathcal{E}$-endowment CBE (S, p), such that $SW(S) \geq SW(S')$. The algorithm runs in polynomial time using* <u>*value*</u> *queries.*

2. [Thm. 7.5] *There exists a polynomial algorithm such that for* <u>*general*</u> *valuations, and every significant endowment effect $\mathcal{E}$ and initial allocation $S' = (S'_1, \ldots, S'_n)$, computes an $\mathcal{E}$-endowment CBE (S, p), such that $SW(S) \geq SW(S')$. The algorithm runs in polynomial time using* <u>*demand*</u> *queries.*

The proof of item 2 in the theorem above implies the following corollary:

[5]This is consistent with the literature on CBE, which has focused on the existence and computation of nearly-efficiency CBEs Dobzinski et al. [2015]; Feldman and Lucier [2014].

Corollary [Corr. 7.6] For every market, and significant endowment effect $\mathcal{E}$, any optimal allocation S can be paired with bundle prices p so that (S, p) is an $\mathcal{E}$-endowment equilibrium.

We note that this result cannot be extended to all endowment effects within our framework. In particular, for endowment effects such that for some $\beta < 1$ it holds that $g^X(X) \leq \beta \cdot v(X)$ for all $X \subseteq M$, there are instances that admit no endowment CBE with optimal welfare, already for XOS valuations (Proposition 7.8). For this subclass of endowment effects, we provide approximation lower bounds as a function of the parameter β, for different classes of valuations (including XOS, subadditive, and arbitrary; see Section 7).

1.4 Comparison to Related Work

Our work builds upon the recent work by Babaioff et al. [2018] that proposed the first formulation for the endowment effect in combinatorial auctions. They show that every market with submodular valuations admits an $\mathcal{E}^I$-endowment equilibrium that gives at least half of the optimal social welfare.

Other relaxations of WE have been considered in the literature in an attempt to ameliorate the non-existence problem of WE, and achieve approximate stability and efficiency for more general valuation classes than gross substitutes.

Fu et al. [2012] considered a relaxed notion of WE, termed *conditional equilibrium*. A conditional equilibrium is a pair of an allocation and item prices satisfying individual rationality, and such that no consumer wishes to expand their allocation, but disposing of items is not allowed. They showed that every conditional equilibrium has at least half of the optimal welfare. Moreover, every market with XOS valuations admits a conditional equilibrium, which can be reached via a "flexible ascent auction", an algorithm proposed by Christodoulou et al. [2016].[6]

A different relaxation of WE was considered by Feldman et al. [2015], where the utility maximization condition is preserved, but market clearance is relaxed (i.e., items with positive prices may be unsold). Using this notion, an equilibrium always exists (say, price all items at some prohibitively large price), but such equilibria carry no approximation guarantees. For this notion it is shown that even for simple markets with two submodular consumers, the social welfare approximation guarantee cannot be better than $\Omega(\sqrt{m})$.

Our results on endowment CBE (Section 7) should be compared with previous notions of bundling equilibria [Feldman and Lucier, 2014; Feldman et al., 2016; Dobzinski et al., 2015]. In these settings, the market designer first partitions the set of items into indivisible bundles $B = \{B_1, \ldots, B_k\}$ (these are the indivisible items in the induced market), and assigns prices to these bundles instead of the original items, and a CBE is a Walrasian equilibrium in the induced market.

Dobzinski et al. [2015] showed that every market (with arbitrary valuations) admits a CBE that gives approximation guarantee of $\tilde{O}(\sqrt{\min\{m, n\}})$. Moreover, given an optimal allocation, a CBE with such approximation can be computed in polynomial time. Furthermore, they provide a polynomial time algorithm that computes a CBE with a $\tilde{O}(m^{2/3})$ approximation guarantee.

This should be compared to Corollary 7.6 and Theorem 7.5 in this paper. Corollary 7.6 shows that for a wide variety of endowment effects (including the one considered by Babaioff *et al.*), there always exists an endowment CBE that gives the *optimal* welfare. Theorem 7.5 shows that for a wide variety of endowment effects (including that of Babaioff *et al.*), given an arbitrary allocation S, one can compute, in polynomial time, an endowment CBE with (weakly) higher welfare than S. Thus, the problem of computing nearly-efficient endowment CBEs is effectively reduced to

[6]Our results imply that their approximation guarantee applies also with respect to the optimal fractional social welfare. (To the best of our knowledge, this was not previously known.

the pure algorithmic problem of welfare approximation — a problem with vast literature (e.g., [Dobzinski et al., 2005; Lehmann et al., 2006; Dobzinski and Schapira, 2006; Feige and Vondrak, 2006; Feige, 2009; Feige and Izsak, 2013; Chakrabarty and Goel, 2010]).

A different notion of bundling equilibria was considered by Feldman et al. [2016]. This notion is a relaxed version of CBE, where some bundles (with positive prices) may remain unsold. Under this notion, for arbitrary valuations, given an arbitrary allocation S, one can compute, in polynomial time, an equilibrium with welfare at least half of the welfare of S.

All the notions above consider a concise set of bundles, a price for each bundle, and an additive pricing over sets of bundles. More general forms of bundle pricing, including non-linear and non-anonymous pricing, lead to welfare-maximizing results, but are highly impractical (in particular, they use an exponential number of prices) [Bikhchandani and Ostroy, 2002; Parkes and Ungar, 2000; Ausubel and Milgrom, 2000; Lahaie and Parkes, 2009; Sun and Yang, 2014].

1.5 Summary

We propose a general principle-based framework for studying the endowment effect in combinatorial markets. We provide both existence and efficiency guarantees of endowment equilibrium (as defined by Babaioff *et al.*) for a wide range of endowment effects and consumer valuation classes. Our main results are: (1) There exist natural endowment effects for which an endowment equilibrium exists for XOS consumers; these equilibria guarantee 2-approximation to the optimal welfare. In contrast, we show that for subadditive consumers, any endowment effect that inflates at a "reasonable" rate does not suffice to guarantee endowment equilibrium existence. (2) For any significant endowment effect, when allowing the seller to pre-pack items into indivisible bundles (thus turning to CBE), given any initial allocation, one can efficiently compute an endowment CBE with (weakly) higher welfare. This result implies that every market admits an optimal endowment CBE. More importantly, it reduces the problem of computing an endowment CBE to the pure algorithmic problem of welfare approximation.

1.6 Paper Organization

Section 2 presents some preliminaries on Walrasian equilibria, valuation classes and query models. The endowment effect framework is described in Section 3.1, followed by Section 3.2 that establishes efficiency guarantees for endowment equilibria, and Section 3.3 which describes the partial order over endowment effects. In Section 4 we provide existence results for endowment equilibria, and in Section 7 we introduce the notion of endowment-Competitive Bundling Equilibrium (CBE), and provide existence, approximation and computational guarantees with respect to this notion. All missing proofs are deferred to the appendix.

2 Preliminaries

Consider a market with a set M of m items and n consumers. Each consumer i has a valuation function $v_i : 2^M \rightarrow \mathbb{R}^+$ that assigns a real value $v_i(X)$ for every subset of items $X \subseteq M$. As standard, assume that valuations are normalized; i.e., $v_i(\emptyset) = 0$, and monotone (free-disposal), i.e., for any $Z \subseteq X$, $v_i(Z) \leq v_i(X)$. An *allocation* is a partition of M to disjoint bundles $S = (S_1, \ldots S_n)$ where bundle S_i is allocated to consumer i.

In this work we measure the quality of an allocation S by its social welfare $SW(S) = \sum_{i \in [n]} v_i(S_i)$. An item pricing is a vector $p = (p_1, \ldots, p_m)$ where p_j is the price of item j. Given an allocation S and item pricing p, consumer i's quasi-linear utility is

$$u_i(S, p) = v_i(S_i) - p(S_i).$$

Given a price vector p and a set X, we use $p(X) = \sum_{j \in X} p_j$.

Definition 2.1. *(Walrasian Equilibrium) A pair of an allocation* $(S_1, \ldots, S_n)$ *and a price vector* $p = (p_1, \ldots p_m)$ *is a Walrasian Equilibrium (WE) if:*

1. **Utility maximization:** *Every consumer receives an allocation that maximizes her utility given the item prices, i.e.,* $v_i(S_i) - \sum_{j \in S_i} p_j \geq v_i(X) - \sum_{j \in X} p_j$ *for every consumer* i *and bundle* $X \subseteq M$.

2. **Market clearance:** *All items are allocated, i.e.,* $\bigcup_{i \in [n]} S_i = M$.

Valuation types. We define the classes of valuation functions considered in this paper, from least to most general, except for unit demand valuations and budget additive valuations which have no containment relation.

- Unit demand: if there exist m values $v^1, \ldots, v^m$, so that $v(X) = \max_{j \in X}\{v^j\}$.
- Submodular: if for any $X, Y \subseteq M$ it holds that $v(X) + v(Y) \geq v(X \cup Y) + v(X \cap Y)$.
- Fractionally subadditive (XOS): if there exist vectors $v_1, \ldots v_k \in \mathbb{R}^M$ so that for any $X \subseteq M$ it holds that $v(X) = max_{i \in [k]} \sum_{j \in X} v_i(j)$.
- Subadditive: if for any $X, Y \subseteq M$ it holds that $v(X) + v(Y) \geq v(X \cup Y)$.

Value and demand queries. The representation of combinatorial valuation functions is exponential in the parameters of the problem. A standard computational model in this setting is an oracle access. We consider two standard oracle models, namely, value and demand oracles:

- A *value query* for valuation v receives a set X as input, and returns $v(X)$.
- A *demand query* for valuation v receives a price vector $p = (p_1, \ldots, p_m)$ as input, and returns a set X that maximizes $u_i(X, p)$.

3 Endowment Effect

3.1 Endowment Effect Framework

In the introduction, we present two principles that underlie the endowment effect, namely the loss aversion principle and the separability principle. The loss aversion principle states that:

$$v^{X \cup Y}(X \cup Y) - v^{X \cup Y}(Y) \geq v^Y(X \cup Y) - v^Y(Y) \qquad \forall X, Y \subseteq M,$$

and the separability principle states that

$$v^X(Y \setminus X \mid X \cap Y) = v(Y \setminus X \mid X \cap Y) \qquad \forall Y \subseteq M.$$

In Lemma A.1 we show that the endowed valuation $v^X : 2^M \to \mathbb{R}^+$ satisfies the separability principle if and only if

$$v^X(Y) = v(Y) + g^X(X \cap Y),$$

for some function $g^X : 2^X \to \mathbb{R}$. In Lemma A.2 we show that the loss aversion principle implies that g^X satisfies $g^X(Z) \leq g^X(X)$ for every $Z \subseteq X$.

For simplicity of presentation, we also assume that the gain functions are normalized; i.e., for all $X \subseteq M$, it holds that $g^X(\emptyset) = 0$. This implies that the endowed valuations are also normalized; i.e., $v^X(\emptyset) = v(\emptyset) + g^X(\emptyset) = 0$. Our results can be generalized to non-normalized gain functions.

Based on this characterization, the following definition follows.

Definition 3.1. *An* endowment effect $\mathcal{E}$ *is a collection of* gain functions $g^X : 2^X \to \mathbb{R}$ *for each* $X \subseteq M$*, such that* $g^X(Z) \leq g^X(X)$ *for all* $Z \subseteq X$*. Given an endowment effect* $\mathcal{E}$*, a valuation function* $v : 2^M \to \mathbb{R}^+$*, and an endowed set* X*, the endowment valuation with respect to* X *is given by*

$$v^{X,\mathcal{E}}(Y) = v(Y) + g^X(X \cap Y)$$

For simplicity, when the endowment effect is clear in the context, we write v^X instead of $v^{X,\mathcal{E}}$. An *endowment environment* is given by a vector of endowment effects for the consumers $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$.

We are now ready to define the notion of endowment equilibrium.

Definition 3.2. *For an instance* $(v_1, \ldots, v_n)$ *and endowment environment* $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$*, a pair* (S, p) *of an allocation* $S = (S_1, \ldots S_n)$ *and a price vector* $p = (p_1, \ldots, p_m)$ *forms an* $\mathcal{E}$*-endowment equilibrium, if* (S, p) *is a Walrasian equilibrium with respect to* $(v_1^{S_1,\mathcal{E}_1}, \ldots, v_n^{S_n,\mathcal{E}_n})$*; i.e.,*

1. **Utility maximization:** *Every consumer receives an allocation that maximizes her endowed utility given the item prices, i.e., for every consumer* i *and bundle* $X \subseteq M$*,*

$$v_i^{S_i,\mathcal{E}_i}(S_i) - \sum_{j \in S_i} p_j \geq v_i^{S_i,\mathcal{E}_i}(X) - \sum_{j \in X} p_j.$$

2. **Market clearance:** *All items are allocated, i.e.,* $\bigcup_{i \in [n]} S_i = M$*.*

We abuse notation and use $\mathcal{E}$ both for endowment effect and endowment environment when all consumers are subject to the same endowment effect.

3.2 Efficiency Guarantees for Endowment Equilibria

Given an endowment environment $\mathcal{E}$, we are interested both in the existence and the social welfare of $\mathcal{E}$-endowment equilibria. Walrasian equilibria are related to the following linear program relaxation for combinatorial auctions, known as the *configuration LP*, (see e.g., [Bikhchandani and Mamer, 1997]). Here, $x_{i,T}$ are the decision variables for every consumer i and set $T \subseteq M$.

Maximize $\sum_{i \in [n]} \sum_{T \subseteq M} x_{i,T} \cdot v_i(T)$
Subject to:

- For each $j \in M$: $\sum_{i \in [n]} \sum_{T \subseteq M | j \in T} x_{i,T} \leq 1$.
- For each $i \in [n]$: $\sum_{T \subseteq M} x_{i,T} \leq 1$.
- For each i, T: $x_{i,T} \geq 0$

The existence of a Walrasian equilibrium turns out to be closely related to the integrality gap of the configuration LP:

Theorem 3.3. *[Nisan and Segal, 2006] An instance* $(v_1, \ldots, v_n)$ *admits a Walrasian Equilibrium if and only if the integrality gap of the configuration LP is* 1*. Moreover, an integral allocation* S *has payments* p *such that* (S, p) *is a Walrasian Equilibrium if and only if* S *is an optimal solution to the LP.*

The following proposition gives an approximation guarantee for every endowment equilibrium, as a function of the gain functions. This is a natural generalization of [Babaioff et al., 2018, Corollary 3.7]. Note that an additional requirement is that the gain functions are non-negative.

Proposition 3.4. *Given an instance $(v_1, \dots, v_n)$, let OPT be the value of the optimal fractional welfare. For an endowment effect $\mathcal{E}$, where $g^X \geq 0$ for all $g^X \in \mathcal{E}$, if (S, p) is an $\mathcal{E}$-endowment equilibrium, then the allocation S has welfare guarantee of at least*

$$\sum_{i\in[n]} v_i(S_i) \geq \frac{\sum_{i\in[n]} v_i(S_i)}{\sum_{i\in[n]}\left(v_i(S_i) + g_i^{S_i}(S_i)\right)} \cdot OPT,$$

where $g_i^{S_i}$ is the gain function corresponding to $Endow_i$.

Proof. Since (S, p) is an $\mathcal{E}$-endowment equilibrium, by Theorem 3.3, for any optimal fractional solution $\{x_{i,T}\}$ of the LP w.r.t. the valuations $(v_1^{S_1,\mathcal{E}_1}, \dots, v_n^{S_n,\mathcal{E}_n})$ it holds that

$$\sum_{i\in[n]} v_i(S_i) + g_i^{S_i}(S_i) = \sum_{i\in[n]} v_i^{S_i,\mathcal{E}_i}(S_i) \geq \sum_{i\in[n]}\sum_{T\subseteq M} x_{i,T}\cdot v_i^{S_i,\mathcal{E}_i}(T) \geq \sum_{i\in[n]}\sum_{T\subseteq M} x_{i,T}\cdot v_i(T) = OPT,$$

where the first inequality is by optimality and the second is by non-negativity of the gain functions. The proof follows by multiplying both sides by $\sum_{i\in[n]} v_i(S_i)$ and rearranging. □

An immediate theorem is the following:

Theorem 3.5. *If (S, p) is an $\mathcal{E}$-endowment equilibrium for instance $(v_1, \dots, v_n)$, and for all v_i it holds that $g_i^{S_i}(S_i) \leq v_i(S_i)$, then the social welfare of S is a 2-approximation to the optimal fractional welfare.*

Theorem 3.5 implies a 2-approximation guarantee for submodular consumers with $\mathcal{E}^{SOM}$ endowment effect, or more generally, with any of the endowment effects listed in the following section.

3.3 Partial order over endowment effects

Recall that an endowment effect $\mathcal{E}$ is specified by a set of gain functions g^X for every set $X \subseteq M$.

We next define a partial order over the set of endowment effects. We discuss each endowment effect $\{g^X\}_X$ through the term $g^X(Z \mid X \setminus Z)$ — the additional loss incurred upon losing a subset Z of an endowed set X due to the endowment effect.

Definition 3.6. *Fix a valuation function v, and two endowment effects $\mathcal{E}, \hat{\mathcal{E}}$ with respect to v.*

- *Given a set X, we write $\mathcal{E} \prec_X \hat{\mathcal{E}}$ if for all $Z \subseteq X$, $g^X(Z \mid X \setminus Z) \leq \hat{g}^X(Z \mid X \setminus Z)$.*
- *We write $\mathcal{E} \prec \hat{\mathcal{E}}$ (and say that $\mathcal{E}$ is* dominated by *$\hat{\mathcal{E}}$, or $\hat{\mathcal{E}}$ dominates $\mathcal{E}$) if for all $X \subseteq M$ it holds that $\mathcal{E} \prec_X \hat{\mathcal{E}}$.*

Add: For example, Identity and Absolute Loss.

The following theorem establishes the stability preservation property of the partial order. In particular, that an endowment effect always preserves the endowment equilibria of endowment effects dominated by it:

Theorem 3.7. *[Stability preservation] Suppose (S, p) is an $\mathcal{E}$-endowment equilibrium with respect to instance $(v_1, \dots, v_n)$, and let $\hat{\mathcal{E}}$ be such that $\mathcal{E}_i \prec_{S_i} \hat{\mathcal{E}}_i$ for every i. Then, (S, p) is also an $\hat{\mathcal{E}}$-endowment equilibrium.*

The proof of Theorem 3.7 is obtained by applying Lemma 3.8 iteratively for each consumer.

Lemma 3.8. *For an instance $(v_1, \ldots, v_n)$, and an endowment environment $\mathcal{E} = (\mathcal{E}_1, \ldots, \mathcal{E}_n)$, let (S, p) be an $\mathcal{E}$-endowment equilibrium. For any consumer i, and endowment effect $\hat{\mathcal{E}}_i$, if $\mathcal{E}_i \prec_{S_i} \hat{\mathcal{E}}_i$, then (S, p) is also an $(\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}, \hat{\mathcal{E}}_i, \mathcal{E}_{i+1}, \ldots, \mathcal{E}_n)$-endowment equilibrium.*

Proof. Let $g_i^{S_i} \in \mathcal{E}_i$, and let $\hat{g}_i^{S_i} \in \hat{\mathcal{E}}_i$. The pair (S, p) is an $\mathcal{E}$-endowment equilibrium, therefore for every $Y \subseteq M$ it holds that

$$v_i(S_i) + g_i^{S_i}(S_i) - p(S_i) \geq v_i(Y) + g_i^{S_i}(S_i \cap Y) - p(Y).$$

Rearranging,

$$v_i(S_i) + g_i^{S_i}(S_i \setminus Y \mid S_i \cap Y) - p(S_i) \geq v_i(Y) - p(Y).$$

Since $\mathcal{E}_i \prec_{S_i} \hat{\mathcal{E}}_i$, by Definition 3.6, the last inequality still holds when $g_i^{S_i}$ is replaced by $\hat{g}_i^{S_i}$. I.e.,

$$v_i(S_i) + \hat{g}_i^{S_i}(S_i \setminus Y \mid S_i \cap Y) - p(S_i) \geq v_i(Y) - p(Y).$$

Rearranging, we conclude that:

$$v_i(S_i) + \hat{g}_i^{S_i}(S_i) - p(S_i) \geq v_i(Y) + \hat{g}_i^{S_i}(S_i \cap Y) - p(Y),$$

i.e., that $v_i^{S_i, \hat{\mathcal{E}}_i}(S_i) - p(S_i) \geq v_i^{S_i, \hat{\mathcal{E}}_i}(Y) - p(Y)$. It follows that S_i maximizes consumer i's utility, as desired. Individual rationality follows by the fact that endowed valuations are normalized. □

4 Existence of Endowment Equilibrium

The main theorem in this section is that for every instance with XOS valuations, there exists an $\mathcal{E}^{AL}$-endowment equilibrium. Moreover, we devise a dynamic process that given an arbitrary initial allocation, terminates in an $\mathcal{E}^{AL}$-endowment equilibrium with at least as much welfare as that of the original allocation.

Theorem 4.1. *There exists a natural dynamic process such that for every market with XOS consumers, and every initial allocation $S' = (S'_1, \ldots, S'_n)$ returns an $\mathcal{E}^{AL}$-endowment equilibrium (S, p), such that $SW(S) \geq SW(S')$.*

Our proof goes through an interesting connection between endowment equilibrium and conditional equilibrium. In Section 4.1 we explore this connection. In Section 4.2 we present the proof of Theorem 4.1.

4.1 Endowment Equilibrium and Conditional Equilibrium

Our analysis draws upon an interesting relation between an endowment equilibrium and a conditional equilibrium [Fu et al., 2012]. The definition of a conditional equilibrium follows.

Definition 4.2. *[Fu et al., 2012] For an instance $(v_1, \ldots, v_n)$, a pair of allocation $(S_1, \ldots, S_n)$ and item pricing $(p_1, \ldots p_m)$ is a conditional equilibrium if for all $i = 1, \ldots, n$,*

1. **Individual rationality:** $\sum_{j \in S_i} p_j \leq v_i(S_i)$
2. **Outward stability:** *For every $X \subseteq M \setminus S_i$, $v_i(X \mid S_i) \leq \sum_{j \in X} p_j$*

We first introduce the notion of *inward stability*. A set X is inward stable if for every set $Y \subseteq M$, the marginal utility of $X \setminus Y$ is non-negative. Formally:

Definition 4.3. *Given a consumer with valuation v, and item pricing $(p_1, \dots p_m)$, a set $X \subseteq M$ is* inward stable *w.r.t. v and p if for every $Y \subseteq M$ it holds that $p(X \setminus Y) \leq v(X \setminus Y \mid Y)$.*

Observation 4.4. *If X is inward stable for consumer i, then there exists a utility-maximizing set of items for consumer i that contains X.*

In general, endowment and conditional equilibria are incomparable notions. The following proposition shows that any endowment equilibrium that is also individually rational with respect to the original valuations is a conditional equilibrium.

Proposition 4.5. *For any instance $(v_1, \dots, v_n)$, if a pair of allocation $(S_1, \dots, S_n)$ and item prices $(p_1, \dots, p_m)$ is an $\mathcal{E}$-endowment equilibrium, and for all consumers i it holds that $p(S_i) \leq v_i(S_i)$, then (S, p) is a conditional equilibrium.*

Proof. Individual rationality $p(S_i) \leq v_i(S_i)$ is given. It remains to show outward stability. For any consumer i with endowment effect $\mathcal{E}_i$, since (S, p) is an endowment equilibrium, it holds that for every $X \subseteq M \setminus S_i$, $v_i^{S_i}(S_i) - p(S_i) \geq v_i^{S_i}(X \cup S_i) - p(X \cup S_i)$, i.e.,

$$g_i^{S_i}(S_i) + v_i(S_i) - p(S_i) \geq g_i^{S_i}(S_i) + v_i(X \cup S_i) - p(X \cup S_i)$$

p is linear so $p(X) \geq v_i(X \mid S_i)$ as required, □

In the other direction, for a conditional equilibrium to be an endowment equilibrium, it needs to be inward stable with respect to the endowed valuations. In the following lemma we give a sufficient condition for inward stability.

Lemma 4.6. *Given a consumer with valuation v, an endowment effect $\{g^X\}_X$, and item pricing $(p_1, \dots, p_m)$, if a set $X \subseteq M$ satisfies*

$$g^X(Z) - p(Z) \leq g^X(X) - p(X) \text{ for all } Z \subseteq X, \tag{4}$$

then X is inward stable with respect to v^X and p.

Proof. Fix any $Y \subseteq M$. By monotonicity of v we have that:

$$v^X(Y) - p(Y) = v(Y) + g^X(X \cap Y) - p(Y) \leq v(X \cup Y) + g^X(X \cap Y) - p(Y)$$

It is given that $g^X(Y \cap X) - p(Y \cap X) \leq g^X(X) - p(X)$. Combining the two inequalities above implies that $v^X(Y) - p(Y) \leq v(X \cup Y) + g^X(X) - p(X) - p(Y \setminus X) = v^X(X \cup Y) - p(X \cup Y)$, as required. □

The following proposition shows that an allocation and prices that are both inward stable with respect to the endowed valuations, and outward stable, form an endowment equilibrium.

Proposition 4.7. *For any instance $(v_1, \dots, v_n)$, if the pair of allocation $(S_1, \dots, S_n)$ and item prices $(p_1, \dots, p_m)$ is a conditional equilibrium, and the endowment environment $\mathcal{E} = (\mathcal{E}_1, \dots, \mathcal{E}_n)$ is such that for every consumer i, the gain function g^{S_i} corresponding to $\mathcal{E}_i$ satisfies*

$$g^{S_i}(Z) - p(Z) \leq g^{S_i}(S_i) - p(S_i) \text{ for all } Z \subseteq S_i, \tag{5}$$

then (S, p) is an $\mathcal{E}$-endowment equilibrium.

Proof. Fix a consumer i and a set $X \subseteq M$. It is given in the proposition that the conditions of Lemma 4.6 on v_i with S_i, $\mathcal{E}_i$ and p hold, therefore, $v_i^{S_i}(X) - p(X) \leq v_i^{S_i}(S_i \cup X) - p(S_i \cup X)$. Since (S, p) is a conditional equilibrium, it holds that $p(X \setminus S_i) - v_i(X \setminus S_i \mid S_i) \geq 0$. It follows that

$$v_i^{S_i}(X) - p(X) \leq v_i(S_i \cup X) + g^{S_i}(S_i) - p(S_i \cup X) + p(X \setminus S_i) - v_i(X \setminus S_i \mid S_i) = v_i^{S_i}(S_i) - p(S_i),$$

therefore, consumer i is utility maximizing. Finally, note that individual rationality follows by considering the case $X = \emptyset$. □

Note that Proposition 4.7 implies that every conditional equilibrium is also an $\mathcal{E}^{AON}$-endowment equilibrium (see Proposition A.3 in Appendix A.2).

4.2 Proof of Main Result (Theorem 4.1)

In this section we show that the $\mathcal{E}^{AL}$-endowment effect leads to strong existence and efficiency guarantees in combinatorial markets with XOS valuations. In particular, we provide a dynamic process (Algorithm 1) that for every market with XOS consumers and initial allocation S, terminates in an $\mathcal{E}^{AL}$-endowment equilibrium with at least as much social welfare as S.[7] An immediate corollary of our proof is that any optimal allocation S can always be paired with prices p such that (S, p) forms an $\mathcal{E}^{AL}$-endowment equilibrium. Since $g^X_{AL}(X) = v(X)$, Theorem 3.5 implies that every $\mathcal{E}^{AL}$-endowment equilibrium gives at least a 2-approximation to the optimal (even fractional) welfare.

We begin by recalling the definition of *supporting prices* [Dobzinski et al., 2005]. Given a valuation v and a set $X \subseteq M$, the prices $\{p_j\}_{j \in X}$ are supporting prices for $v(X)$ if $v(X) = \sum_{j \in X} p_j$ and for every $Z \subseteq X$, $v(Z) \geq \sum_{j \in Z} p_j$. A valuation is XOS if and only if for all $X \subseteq M$ there exist supporting prices for $v(X)$ (see, e.g., [Dobzinski et al., 2005]).

The following lemma shows that for XOS valuations, the condition of Lemma 4.6 holds with respect to the endowment effect $\mathcal{E}^{AL}$, and a set of supporting prices.

Lemma 4.8. *Fix a consumer with an XOS valuation v. If $(p_1, \ldots, p_m)$ are supporting prices w.r.t. v and $X \subseteq M$, then the gain function g^X corresponding to $\mathcal{E}^{AL}$ satisfies $g^X(Z) - p(Z) \leq g^X(X) - p(X)$ for all $Z \subseteq X$.*

Proof. Observe that by definition of supporting prices, it holds that $p(X \setminus Z) \leq v(X \setminus Z)$ for any $Z \subseteq X$. By definition of $g^X \in \mathcal{E}^{AL}$, we have that $g^X(Z) = v(X) - v(X \setminus Z)$. Rearranging, we conclude that

$$g^X(X) - g^X(Z) = v(X) - g^X(Z) = v(X \setminus Z) \geq p(X \setminus Z) = p(X) - p(Z),$$

as required. □

The above lemma has an immediate implication. In [Fu et al., 2012] it was shown that in an instance $(v_1, \ldots, v_n)$ of XOS valuations, for any welfare-maximizing allocation S, if one sets the prices of items p in each S_i to be the supporting prices with respect to v_i and S_i, then (S, p) is a conditional equilibrium. Combining the last observation with Lemma 4.8 and Proposition 4.7, we conclude the following:

Corollary 4.9. *For every market with XOS consumers, every optimal allocation S can be paired with item prices p so that (S, p) is an $\mathcal{E}^{AL}$-endowment equilibrium.*

[7] Algorithm 1 is a modified version of the "flexible ascent auction" presented by Fu et al. [2012].

We now show that given a starting allocation S', one can run a modified version of the "flexible ascent auction" from [Fu et al., 2012], that results in an $\mathcal{E}^{AL}$-endowment equilibrium (S,p) with at least as much welfare as S'. Moreover, (S,p) is a conditional equilibrium, and $SW(S)$ is at least $\max\{\frac{OPT}{2}, SW(S')\}$ (see Proposition 3.4).

That (S,p) is a conditional equilibrium follows by Proposition 4.5 and the fact that (S,p) satisfies individual rationality with respect to the original valuations.

ALGORITHM 1: An $\mathcal{E}^{AL}$-endowment flexible ascent auction for XOS valuations.

Input: XOS valuations $(v_1,\ldots,v_n)$, allocation $(S'_1,\ldots,S'_n)$.;
Output: Allocation $S_1,\ldots,S_n$, prices $p_1,\ldots,p_m$
Set $S \leftarrow S'$
Set $p_1,\ldots,p_m$ such that for all $i \in [n]$ the prices $\{p_j \mid j \in S_i\}$ are supporting prices for S_i w.r.t. v_i
while $\exists i, X \subseteq M$ *such that* $v_i^{S_i}(X) - p(X) > v_i^{S_i}(S_i) - p(S_i)$ **do**
  $S_i = S_i \cup X$
  $S_j = S_j \setminus X \quad \forall j \neq i$
  Set $p_1,\ldots,p_m$ such that for all $i \in [n]$ the prices $\{p_j \mid j \in S_i\}$ are supporting prices for S_i w.r.t. v_i
end
return (S,p)

The main difference of Algorithm 1 compared to the flexible ascent auction is that in the end of every iteration all the prices may change, not only the ones demanded in the current iteration. Specifically, given that at some iteration consumer i, who was previously allocated S_i, is now allocated $S_i \cup X$ for some X, then for all j such that $S_j \setminus X \subsetneq S_j$, the prices of $S_j \setminus X$ may change, so that the prices are supporting prices with respect to every consumer, and thus inward stability is maintained. This property implies that restricting attention to deviations that take the form of extending the current allocation is without loss.

The following lemma shows that the dynamics in Algorithm 1 are better-response dynamics.

Lemma 4.10. *Let S and p be the allocation and price vector at the beginning of some iteration in Algorithm 1. For the chosen consumer i, and her corresponding set X, it holds that $v_i^{S_i}(S_i \cup X) - p(S_i \cup X) > v_i^{S_i}(S_i) - p(S_i)$. I.e., consumer i performs a beneficial deviation.*

Proof. At the end of every iteration, all prices are adjusted to be supporting prices for every consumer. By chaining Lemma 4.6 and Lemma 4.8, we conclude that the allocation of every consumer is inward stable. Therefore, $v_i^{S_i}(S_i \cup X) - p(S_i \cup X) \geq v_i^{S_i}(X) - p(X) > v_i^{S_i}(S_i) - p(S_i)$, where the first inequality follows by inward stability, and the second inequality follows by the design of the algorithm. □

We next conclude that the welfare strictly increases as the algorithms progresses.

Proposition 4.11. *At each iteration of Algorithm 1, the social welfare strictly increases.*

Proof. At the beginning of every iteration, for every consumer i, the prices p are supporting prices with respect to S_i and v_i. Therefore, $p(S_j) = v_j(S_j)$ for all $j \in [n]$, which implies that $p(M) = \sum_j v_j(S_j)$. Let i be the chosen consumer at the current iteration, and X be her corresponding set according to the algorithm. Let S^{new} be the allocation obtained at the end of the iteration. Then

$$\sum_{j\in[n]} v_j(S_j^{new}) = v_i(S_i \cup X) + \sum_{j\neq i} v_j(S_j \setminus X) = v_i(S_i) + v_i(X \setminus S_i \mid S_i) + \sum_{j\neq i} v_j(S_j \setminus X). \quad (6)$$

Lemma 4.10 together with Equation (3) gives that $v_i(X \setminus S_i \mid S_i) = v_i^{S_i}(X \setminus S_i \mid S_i) > p(X \setminus S_i)$. Combined with Equation (6) we get

$$\sum_{j\in[n]} v_j(S_j^{new}) > p(S_i)+p(X\setminus S_i)+\sum_{j\neq i} v_j(S_j\setminus X) \geq p(S_i)+p(X\setminus S_i)+\sum_{j\neq i} p(S_j\setminus X) = p(M) = \sum_j v_j(S_j).$$

□

Proposition 4.11 implies that the algorithm terminates. The following proposition shows it terminates at a conditional equilibrium.

Proposition 4.12. *When Algorithm 1 terminates at allocation S and price vector p, (S,p) is a conditional equilibrium.*

Proof. When the algorithm terminates, by definition of the condition in the while loop, it holds that for all i and $X \subseteq M \setminus S_i$,

$$v_i^{S_i}(X \cup S_i) - p(X \cup S_i) \leq v_i^{S_i}(S_i) - p(S_i)$$

Rearranging, it follows that $v_i^{S_i}(X \mid S_i) \leq p(X \setminus S_i)$. Combining with Equation (3) (separability) implies outward stability. Individual rationality for each consumer follows by the fact that the prices are supporting prices for each consumer. □

With this, we have all the components needed to conclude the proof of Theorem 4.1.

Proof of Theorem 4.1. By Proposition 4.12 initiating the algorithm at S' terminates at a conditional equilibrium (S,p), such that (by Proposition 4.11) $SW(S) \geq SW(S')$. Chaining Propositions 4.8 and Proposition 4.7, we get that (S,p) is an $\mathcal{E}^{AL}$-endowment equilibrium. □

5 Sum-of-Marginals (SOM) Endowment Effect

In this section we introduce a new endowment effect, called *Sum of Marginals*, denoted $\mathcal{E}^{SOM}$. The gain function of the $\mathcal{E}^{SOM}$ endowment effect given an endowment X is given by

$$g_{SOM}^X(Z) = \sum_{j\in Z} v(j \mid X \setminus j).$$

add $g(X \mid X \setminus Z)$ and intuition.

The main theorem of this section is Theorem 5.2, showing that for submodular consumers there always exists an $\mathcal{E}^{SOM}$-endowment equilibrium that gives 2-approximation to the optimal welfare. Recall that Babaioff *et al.* establish the same result with respect to the Identity endowment effect. Proposition 5.1 shows that $\mathcal{E}^{SOM}$ is strictly weaker than $\mathcal{E}^I$, implying that Theorem 5.2 strengthens the main result of Babaioff *et al.*.

Proposition 5.1. *For every submodular valuation v, it holds that $\mathcal{E}^{SOM} \prec \mathcal{E}^I$.*

Proof. Fix a set $X \subseteq M$, and let $g_{SOM}^X \in \mathcal{E}^{SOM}$ and $g_I^X \in \mathcal{E}^I$. For all $Z \subseteq X$ we need to show that $g_{SOM}^X(Z \mid X \setminus Z) \leq g_I^X(Z \mid X \setminus Z)$. By the additivity of g_{SOM}^X, it follows that $g_{SOM}^X(Z \mid X \setminus Z) = g_{SOM}^X(Z)$. Therefore, it remains to show that $g_{SOM}^X(Z) \leq g_I^X(Z \mid X \setminus Z)$.

Rename the items in Z by $1, \ldots, |Z|$, and let Z_j denote the set of items $\{1, \ldots, j\}$. It holds that

$$g_{SOM}^X(Z) = \sum_{j\in Z} v(j \mid X \setminus \{j\}) \leq \sum_{j\in Z} v(j \mid X \setminus Z_j) = v(Z \mid X \setminus Z) = g_I^X(Z \mid X \setminus Z),$$

where the inequality holds by submodularity, and the last equality holds by definition of g_I^X. □

show that SOM and Identity are far from each other.
Add: approximation results (both to statement and to proof), and organize.

Theorem 5.2. *Let $(v_1, \ldots, v_n)$ be an instance of submodular valuations. There exists an allocation $S = (S_1, \ldots, S_n)$ and item prices $p = (p_1, \ldots, p_m)$ so that (S, p) is an $\mathcal{E}^{SOM}$-endowment equilibrium.*

In our proof, we adjust the techniques of Babaioff *et al.* to our framework, and show that their techniques essentially apply also to the weaker Sum-of-Marginals endowment effect, leading to a stronger result.

We begin with the definition of local optimum.

Definition 5.3. *[Babaioff et al., 2018] For an instance $(v_1, \ldots, v_n)$, an allocation $(S_1, \ldots, S_n)$ is a local optimum if $\cup_{i\in[n]} S_i = M$ and for every pair of consumers $i, i' \in [n]$ and item $j \in S_i$ it holds that $v_i(S_i) + v_{i'}(S_{i'}) \geq v_i(S_i \setminus \{j\}) + v_{i'}(S_{i'} \cup \{j\})$*

The following proposition (essentially [Babaioff et al., 2018, Claim 4.4] combined with individual rationality) shows that for submodular valuations, the $\mathcal{E}^I$-endowment equilibria suggested by Babaioff *et al.* are also conditional equilibria.

Proposition 5.4. *Let $(v_1, \ldots, v_n)$ be an instance of submodular valuations, $S = (S_1, \ldots, S_n)$ be some locally optimal allocation, and $p = (p_1, \ldots, p_m)$ be item prices defined by $p_j = v_{i(j)}(j \mid S_{i(j)} \setminus \{j\})$, where $i(j)$ is the consumer i such that $j \in S_i$. Then, (S, p) is a conditional equilibrium.*

Proof. Individual rationality: fix consumer i, and order the items in S_i in some order $1, 2, \ldots, |S_i|$, then $v_i(S_i) = \sum_{j\in S_i} v_i(j \mid \{1, \ldots j-1\}) \geq \sum_{j\in S_i} v_i(j \mid S_i \setminus \{j\}) = p(S_i)$, where the inequality follows by submodularity.

Outward stability: fix a consumer i and consider some $X \subseteq M \setminus S_i$. Since S is a local optimum, for every $j \in X$ it holds that $v_i(j \mid S_i) \leq p_j = v_{i(j)}(j \mid S_{i(j)} \setminus \{j\})$. Order the items in X in some order $1, 2, \ldots, |X|$, then

$$v_i(X \mid S_i) = \sum_{j\in X} v_i(j \mid S_i \cup \{1, \ldots j-1\}) \leq \sum_{j\in X} v_i(j \mid S_i) \leq \sum_{j\in X} p_j$$

where the first inequality follows by submodularity. □

The proof of Theorem 5.2 now follows by the definition of the endowment effect $\mathcal{E}^{SOM}$.

Proof of Theorem 5.2. Consider a locally optimal allocation S and the prices $p_j = v_{i(j)}(j \mid S_{i(j)} \setminus \{j\})$ where $i(j)$ is the consumer i such that $j \in S_i$. By Lemma 5.4, it holds that (S, p) is a conditional equilibrium. Moreover, for every consumer i, and for every $Z \subseteq S_i$, it holds that

$$g^{S_i}(Z) - p(Z) = \sum_{j\in Z} v(\{j\} \mid S_i \setminus \{j\}) - \sum_{j\in Z} v(\{j\} \mid S_i \setminus \{j\}) = 0 = g^{S_i}(S_i) - p(S_i).$$

Thus, by Proposition 4.7, (S, p) is an $\mathcal{E}^{SOM}$-endowment equilibrium. add approximation proof. □

6 Beyond XOS Valuations

So far we've shown that the endowment effect can be harnessed to stabilize settings more general than gross-substitutes, in particular up to XOS valuations. Can we harness stability via the endowment effect further? Without any reasonable restriction on the endowment effect, this question can be answered affirmatively fairly easily. Specifically, for an endowment effect that inflates the value of a set linearly in the number of items, we show that an endowment equilibrium always exists. This result has a similar flavor to the observation made by Babaioff et al. [2018, Proposition 3.4], showing that for any instance there exists a sufficiently large α such that en $\alpha \cdot \mathcal{E}^I$-endowment equilibrium always exists. Yet, while the value of α required for their result depends on the valuations of other consumers, our endowment effect is simpler, and does not depends on others' valuations. write the above sentences better.

handle $i(j)$. consider replacing by i_j and moving out of theorems.

Proposition 6.1. *Let $\mathcal{E}^{PROP} = \{g^X(Z) = |Z| \cdot v(X) : X \subseteq M\}$. Let $(v_1, \ldots, v_n)$ be an arbitrary instance of valuations, $S = (S_1, \ldots, S_n)$ be an optimal allocation, and $p = (p_1, \ldots, p_m)$ be item prices, such that $p_j = v_{i(j)}(S_{i(j)})$, where $i(j)$ is the consumer i such that $j \in S_i$. Then, (S, p) is an $\mathcal{E}^{PROP}$-endowment equilibrium.*

change X to Y in the following proof

Proof. W.l.o.g., all items are allocated in S. We need to show that for every i and X it holds that $v_i^{S_i}(S_i) - p(S_i) \geq v_i^{S_i}(X) - p(X)$.

First note that by monotonicity of v_i, it holds that

$$\begin{aligned}
v_i^{S_i}(X) - p(X) &= v_i(X) + |X \cap S_i| \cdot v_i(S_i) - p(X \setminus S_i) - |X \cap S_i| \cdot v_i(S_i) \\
&= v_i(X) - p(X \setminus S_i) \\
&\leq v_i(X \cup S_i) - p(X \setminus S_i) \qquad (7)
\end{aligned}$$

Let $i(j)$ denote the consumer i for which $j \in S_i$, then

$$\begin{aligned}
v_i(X \cup S_i) - p(X \setminus S_i) &= v_i(S_i) + v_i(X \setminus S_i \mid S_i) - \sum_{j \in X \setminus S_i} v_{i(j)}(S_{i(j)}) \\
&= v_i(S_i) + v_i(X \setminus S_i \mid S_i) - \sum_{i' \neq i} |X \cap S_{i'}| \cdot v_{i'}(S_{i'}) \\
&\leq v_i(S_i) + v_i(X \setminus S_i \mid S_i) - \sum_{i' \neq i} |X \cap S_{i'}| \cdot v_{i'}(S_{i'} \cap X \mid S_{i'} \setminus X) \\
&\leq v_i(S_i) + v_i(X \setminus S_i \mid S_i) - \sum_{i' \neq i} v_{i'}(S_{i'} \cap X \mid S_{i'} \setminus X) \qquad (8)
\end{aligned}$$

where the first inequality follows by monotonicity, and the second inequality follows since equality holds whenever $|X \cap S_{i'}| \leq 1$, and strict inequality holds otherwise.

Since S is an optimal allocation, it holds that $v_i(X \setminus S_i \mid S_i) - \sum_{i' \neq i} v_{i'}(S_{i'} \cap X \mid S_{i'} \setminus X) \leq 0$, otherwise reallocating $X \setminus S_i$ to consumer i strictly increases the welfare. Combining with Inequalities (7) and (8), we conclude that

$$v_i^{S_i}(X) - p(X) \leq v_i(S_i) = v_i^{S_i}(S_i) - p(S_i),$$

as required, where the last equality follows since $p(S_i) = g^{S_i}(S_i)$. □

We now show that for subadditive valuations, and endowment effects that inflate valuations by a "reasonable" amount, an endowment equilibrium may not exist.

Proposition 6.2. *For any number of items $m \geq 3$, there exists an instance with identical items, one subadditive consumer and one unit demand consumer, such that for any $\beta \leq O(\sqrt{m})$, and any endowment environment $\mathcal{E}$ that satisfies $g_i^X(X) \leq \beta \cdot v_i(X)$, for every consumer i and $g_i^X \in \mathcal{E}_i$, no $\mathcal{E}$-endowment equilibrium exists.*

Proof. Consider the following instance. Consumer 1 is subadditive with valuation $v_1([m]) = 2$, $v_1(\emptyset) = 0$, and $v_1(X) = 1$ otherwise. Consumer 2 is unit demand with valuation $v_2(X) = \sqrt{\frac{2}{m}}$ for all $\emptyset \neq X \subseteq [m]$. For any β satisfying $m > 2(\beta + 1)^2$, consider an allocation where v_1 gets all items,

$$v_1^{[m]}([m]) = g_1^{[m]}([m]) + v_1([m]) \leq \beta \cdot 2 + 2,$$

where the inequality follow by the assumption of the proposition. By individual rationality, it must be that $v_1^{[m]}([m]) - p([m]) \geq 0$ therefore there exists an item $j \in [m]$ such that $p_j \leq \frac{2(\beta+1)}{m}$. But then consumer 2 is not utility maximizing, because:

$$v_2^{\emptyset}(\{j\}) - p_j = \sqrt{\frac{2}{m}} - p_j \geq \sqrt{\frac{2}{m}} - \frac{2(\beta+1)}{m} > 0,$$

where the last inequality follows by the restriction on β.

Alternatively, consider an allocation where consumer 2 is allocated a non-empty set X, then her value in the endowed valuation is

$$v_2^X(X) = g_2^X(X) + v_2(X) \leq (\beta + 1) \cdot v_2(X) = (\beta + 1) \cdot \sqrt{\frac{2}{m}} < 1,$$

where the last inequality follows by the restriction on β. On the other hand, the marginal contribution of set X to consumer 1 is at least 1. Therefore, this cannot be an endowment equilibrium, since it is sub-optimal with respect to the endowed valuations. □

To summarize, Proposition 6.1 shows that the endowment effect $\mathcal{E}^{PROP}$, which inflates the valuation by a factor of $O(m)$ guarantees existence of endowment equilibrium (and even endowment equilibria with an optimal allocation). Proposition 6.2 shows that inflating the valuation by a factor of $O(\sqrt{m})$ does not suffice for guaranteeing existence in general. Closing this gap is an interesting open problem.

Note that endowment effects $\mathcal{E}^I$ and $\mathcal{E}^{AL}$ inflate the valuation by a factor of 2.

7 Bundling

In this section we study the role of bundling in market efficiency and stability. We assume that the market designer partitions the set of items into indivisible bundles, and these bundles are the items in the induced market. We show that under a wide variety of endowment effects, the bundling operation can recover stability and maintain efficiency.

A bundling $B = \{B_1, \ldots, B_k\}$ is a partition of the set of items M into k disjoint bundles ($\cup_{j \in [k]} B_k = M$). When clear in the context, given a set of indices $T \subseteq \{1, \ldots, k\}$, we slightly abuse notation and write T to mean $\cup_{j \in T} B_j$.

The notion of *competitive bundling equilibrium* is introduced in Dobzinski et al. [2015]:

Definition 7.1. *[Dobzinski et al., 2015] A Competitive Bundling Equilibrium (CBE) is a bundling $B = \{B_1, \ldots, B_k\}$ of M, a pair (S, p) of an allocation $S = (S_1, \ldots, S_n)$ of the bundles to consumers together with bundle prices $p = (p_1, \ldots, p_k)$ such that:*

1. **Utility maximization:** *Every consumer receives an allocation that maximizes her utility given the bundle prices, i.e., for every consumer i and subset of bundles indexed by $T \subseteq [k]$, $v_i(S_i) - \sum_{j \in S_i} p_j \geq v_i(T) - \sum_{j \in T} p_j$*
2. **Market clearance:** *All items are allocated, i.e., $\bigcup_{i \in [n]} \cup_{j \in S_i} B_j = M$.*

The natural combination of CBE and $\mathcal{E}$-endowment equilibrium is simply a CBE with respect to the valuations subject to the endowment environment $\mathcal{E}$. Here again, given a bundling B, and a set of bundles T, we abuse notation and write $v^{T,\mathcal{E}}$ to denote $v^{(\cup_{j\in T}B_j),\mathcal{E}}$.

Definition 7.2. *($\mathcal{E}$-endowment CBE) An $\mathcal{E}$-endowment Competitive Bundling Equilibrium (CBE) is a bundling $B = \{B_1, \ldots, B_k\}$ of M, a pair (S, p) of an allocation $S = (S_1, \ldots, S_n)$ of the bundles to consumers together with bundle prices $(p_1, \ldots, p_k)$ such that:*

1. **Utility maximization:** *Every consumer receives an allocation that maximizes her endowed utility given the bundle prices, i.e., for every consumer i and subset of bundles indexed by $T \subseteq [k]$, $v_i^{S_i,\mathcal{E}_i}(S_i) - \sum_{j\in S_i} p_j \geq v_i^{S_i,\mathcal{E}_i}(T) - \sum_{j\in T} p_j$.*
2. **Market Clearance:** *All items are allocated, i.e., $\bigcup_{i\in[n]} \cup_{j\in S_i} B_j = M$.*

When clear in the context we abuse notation and specify an $\mathcal{E}$-endowment CBE by a pair (S, p) of an allocation S and pricing p. When doing so, we implicitly assume that the bundling is $B = \{S_1, \ldots, S_n\}$.

Demand queries in reduced markets. Consider the market induced by bundling $\{B_1, \ldots, B_k\}$. Given a valuation v and a price vector $p = (p_1, \ldots, p_k)$, a *demand query* returns a set of bundles in $\arg\max_{T\subseteq[k]} v_i(T) - \sum_{j\in T} p_j$.

Our results in this section apply to endowment environments that consist of *significant* endowment effects.

Definition 7.3. *An endowment effect $\mathcal{E}_i$ is* significant *if for every $X \subseteq M$, it holds that $g_i^X(X) \geq v_i(X)$, where g_i^X is the gain function corresponding to $\mathcal{E}_i$.*

For example, the endowment effect considered in Babaioff *et al.*($\mathcal{E}^I$) is significant, as well as the absolute loss ($\mathcal{E}^{AL}$).

Our main results in this section are the following:

Theorem 7.4. *There exists an algorithm such that for <u>submodular</u> valuations, and every significant endowment effect $\mathcal{E}$ and initial allocation $S' = (S'_1, \ldots, S'_n)$ computes an $\mathcal{E}$-endowment CBE (S, p), such that $SW(S) \geq SW(S')$. The algorithm runs in polynomial time using <u>value</u> queries.*

Theorem 7.5. *There exists an algorithm such that for <u>general</u> valuations, and every significant endowment effect $\mathcal{E}$ and initial allocation $S' = (S'_1, \ldots, S'_n)$ computes an $\mathcal{E}$-endowment CBE (S, p), such that $SW(S) \geq SW(S')$. The algorithm runs in polynomial time using <u>demand</u> queries.*

As a corollary of the proof of Theorem 7.5, we show that any optimal allocation can be paired with bundle prices to form an $\mathcal{E}$-endowment CBE.

Corollary 7.6. *For every market, and significant endowment effect $\mathcal{E}$, any optimal allocation S can be paired with bundle prices p so that (S, p) is an $\mathcal{E}$-endowment equilibrium.*

7.1 Computation of Approximately-Optimal Endowment CBEs

In this section we give a black-box reduction from welfare approximation in an endowment CBE to the pure algorithmic problem of welfare approximation. In particular, we show that for any welfare approximation algorithm ALG, and any significant endowment environment $\mathcal{E}$, there exists an algorithm that computes an $\mathcal{E}$-endowment CBE with the same approximation guarantee of ALG.

For submodular valuations, this reduction makes a polynomial number of value queries. For general valuations, it makes a polynomial number of demand queries.[8]

[8]The demand queries required are with respect to any induced market along the process.

We begin by showing that given an allocation $(S_1, \dots, S_n)$, and any endowment environment $\mathcal{E}$, for the bundling $B = \{S_1, \dots, S_n\}$, and the allocation S, together with prices $p_i \le g_i^{S_i}(S_i)$, no consumer i gains by discarding S_i.

Lemma 7.7. *For any instance $(v_1, \dots, v_n)$, allocation $S = (S_1, \dots, S_n)$, endowment environment $\mathcal{E}$, and prices satisfying $p_i \le g_i^{S_i}(S_i)$ for all i, it holds that for all i and $A \subseteq [n] \setminus \{i\}$,*

$$v_i^{S_i,\mathcal{E}_i}(\cup_{k\in A} S_k) - \sum_{\cup_{k\in A}} p_k \le v_i^{S_i,\mathcal{E}_i}(\cup_{k\in A\cup\{i\}} S_k) - \sum_{\cup_{k\in A\cup\{i\}}} p_k$$

Proof. The endowed utility of consumer i from $\cup_{k\in A\cup\{i\}} S_k$ is

$$v_i^{S_i,\mathcal{E}_i}(\cup_{k\in A\cup\{i\}} S_k) - \sum_{k\in A\cup\{i\}} p_k = g_i^{S_i}(S_i) + v_i(\cup_{k\in A\cup\{i\}} S_k) - \sum_{k\in A\cup\{i\}} p_k$$

Since $p_i \le g_i^{S_i}(S_i)$ the above is at least

$$v_i(\cup_{k\in A\cup\{i\}} S_k) - \sum_{k\in A} p_k \ge v_i(\cup_{k\in A} S_k) - \sum_{k\in A} p_k = v_i^{S_i,\mathcal{E}_i}(\cup_{k\in A} S_k) - \sum_{k\in A} p_k$$

where the inequality is by monotonicity and the last equality is since $i \notin A$. □

ALGORITHM 2: An algorithm for computing an $\mathcal{E}$-endowment CBE for submodular valuations

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Input: Allocation $S_1, \dots, S_n$, submodular valuations $v_1, \dots, v_n$.;
Output: Allocation $S_1, \dots, S_n$, prices $p_1, \dots, p_n$.
while *true* **do**
  if $\exists i, j \in [n]$ *so that* $v_i(S_j|S_i) > v_j(S_j)$ **then**
    $S_i \leftarrow S_i \cup S_j$
    $S_j \leftarrow \emptyset$.
  end
  else
    return $(S_1, \dots, S_n)$, $p = (v_1(S_1), \dots, v_n(S_n))$
  end
end

Change reductions from S to S' rather than vice versa. throughout.

We are now ready to present the proofs of Theorem 7.4 (submodular valuations) and Theorem 7.5 (general valuations). We begin with the proof of Theorem 7.4.

Update this proof. Note that the alg may need to compare more than $2n^2$ bundles, since it may nee to query bundles that did not end up merging. Maybe forget about amortized analysis altogether.

Proof of Theorem 7.4. We first claim that the social welfare strictly increases in every iteration of the while loop. To see this, suppose consumers i, j are chosen in some iteration. Then, consumer i is allocated $S_i \cup S_j$, and consumer j is left with nothing. By the design of the algorithm, this only happens if $v_i(S_i \cup S_j) > v_i(S_i) + v_j(S_j)$. Therefore, the total value of consumers i and j strictly increased. Since other consumers' allocations did not change, the social welfare strictly increases.

We now prove that the algorithm runs in $poly(n)$ time, using $poly(n)$ value queries.

The algorithm begins with n bundles, and bundles only merge throughout the algorithm. Thus, there can be at most $2n$ different bundles throughout the algorithm. Since n consumers are queried, the algorithm uses at most $2n^2$ value queries throughout.

Moreover, the algorithm runs in $O(n^4)$ time: each iteration requires $O(n^2)$ time, and a specific bundle cannot be allocated to a specific consumer more than once. Since there are n consumers and at most $2n$ different possible bundles, and each iteration either transfers a bundle from one consumer to another, or merges two bundles, there will be at most $2n^2$ iterations.

Let S be the outcome of Algorithm 2. It remains to show that whenever allocation S satisfies

$$v_i(S_j|S_i) \leq v_j(S_j) \text{ for all } i,j \in [n], \tag{9}$$

the prices $p_i = v_i(S_i)$ set by the algorithm together with the allocation S form an $\mathcal{E}$-endowment CBE[9].

The endowed utility of each consumer i in the outcome (S,p) is $g_i^{S_i}(S_i)$. Suppose by contradiction that some consumer i is not (endowed) utility maximizing. Then, there exists a set $A \subseteq [n]$ so that i would prefer taking the bundles indexed by A, i.e.,

$$\begin{aligned} g_i^{S_i}(S_i) < v_i^{S_i}(\cup_{j\in A} S_j) - \sum_{j\in A} p_j \leq v_i^{S_i}(\cup_{j\in A\cup\{i\}} S_j) - \sum_{j\in A\cup\{i\}} p_j \\ = g_i^{S_i}(S_i) + v_i(\cup_{j\in A\cup\{i\}} S_j) - \sum_{j\in A\cup\{i\}} v_j(S_j) \end{aligned}$$

where the second inequality is by Lemma 7.7, which holds since $\mathcal{E}_i$ is significant. Suppose A is ordered in some arbitrary way and denote by $A_{<j}$ all the elements in A that precede the j-th bundle in A. Then by cancelling out $g_i^{S_i}(S_i)$, the above inequality can be rewritten as

$$0 < v_i(\cup_{j\in A} S_j|S_i) - \sum_{j\in A\setminus\{i\}} v_j(S_j) = \sum_{j\in A\setminus\{i\}} v_i(S_j| \cup_{k\in\{i\}\cup A_{<j}} S_k) - v_j(S_j) \leq \sum_{j\in A\setminus\{i\}} v_i(S_j|S_i) - v_j(S_j),$$

where the last inequality follows by submodularity. Therefore, at least one summand in the right-hand-side expression is positive, which contradicts (9). □

The proof of Theorem 7.4 shows that for submodular valuations, it suffices to check in each iteration the marginal contribution of a single bundle. For more general valuations, this is not sufficient. However, the following theorem shows that the same type of reduction can be obtained for general valuations, using demand queries.

Proof of Theorem 7.5. We claim that at each while iteration the welfare increases. Suppose at a current iteration consumer i is re-allocated $\cup_{j\in A\cup\{i\}} S_j$, and consumers in $A\setminus\{i\}$ are allocated the empty set. By definition of the algorithm this happens only if $v_i(\cup_{j\in A} S_j) - \sum_{j\in A} p_j > v_i(S_i) - 0$.

By monotonicity of v_i, it holds that $v_i(\cup_{j\in A\cup\{i\}} S_j) \geq v_i(\cup_{j\in A} S_j)$, and by the way the prices are defined in the algorithm (that is, $p_j = v_j(S_j)$ for all $j \neq i$, and $p_i = 0$), it follows that $v_i(\cup_{j\in A\cup\{i\}} S_j) - \sum_{j\in A\cup\{i\}} v_j(S_j) > 0$. This difference is exactly the change in social welfare due to the re-allocation of $\cup_{j\in A\cup\{i\}} S_j$ to consumer i (and all $j \in A \setminus \{i\}$ are allocated the empty set). Since the allocation to consumers outside of $A \cup \{i\}$ did not change, the social welfare increases.

As in the proof of Theorem 7.4 the number of different bundles is at most $2n$, and there are n consumers, therefore there are at most $2n^2$ demand queries. Furthermore, as in the proof of Theorem 7.4, the number of while iterations is at most $O(n^2)$, hence the algorithm runs in polynomial time with a polynomial number of demand queries.

Let (S,p) be the outcome of Algorithm 3 (recall that $p_i = v_i(S_i)$.[10]). For every i and $A \subseteq [n]$ it holds that

$$v_i(S_i) \geq v_i(\cup_{j\in A\cup\{i\}} S_j) - \sum_{j\in A\setminus\{i\}} v_j(S_j). \tag{10}$$

[9]An almost identical proof shows that the prices $p_i = g_i^{S_i}(S_i)$ produces the same result.
[10]An almost identical proof shows that the prices $p_i = g_i^{S_i}(S_i)$ also suffice.

ALGORITHM 3: An algorithm for significant $\mathcal{E}$-endowment CBE for general valuations

Input: Allocation $(S_1, \ldots, S_n)$, valuation functions $(v_1, \ldots, v_n)$.;
Output: Allocation $(S_1, \ldots, S_n)$, prices $(p_1, \ldots, p_n)$.
flag = True
while *flag* **do**
  flag = False
  for $i = 1, \ldots, n$ **do**
    $p_i = 0, \quad p_j = v_j(S_j), \forall j \neq i$
    $A \leftarrow \arg\max_{S \subseteq [n]}(v(S) - \sum_{j \in S} p_j)$
    if $v_i(A) - \sum_{j \in A} p_j > v_i(S_i)$ **then**
      $S_i \leftarrow S_i \cup (\cup_{j \in A} S_j)$
      $S_j \leftarrow \emptyset \quad \forall j \in A \setminus \{i\}$.
      flag = True
    end
  end
end
return $(S_1, \ldots, S_n), p = (v_1(S_1), \ldots, v_n(S_n))$.

The utility of each consumer in (S, p) is $g_i^{S_i}(S_i)$. Suppose by contradiction that some consumer i is not utility maximizing, then there exists a set $A \subseteq [n]$ so that

$$\begin{aligned} g_i^{S_i}(S_i) < v_i^{S_i,\mathcal{E}_i}(\cup_{j \in A} S_j) - \sum_{j \in A} p_j \leq v_i^{S_i,\mathcal{E}_i}(\cup_{j \in A \cup \{i\}} S_j) - \sum_{j \in A \cup \{i\}} p_j \\ = g_i^{S_i}(S_i) + v_i(\cup_{j \in A \cup \{i\}} S_j) - \sum_{j \in A \cup \{i\}} v_j(S_j), \end{aligned}$$

where the second inequality follows by Lemma 7.7, which holds since $\mathcal{E}_i$ is significant. By cancelling out $g_i^{S_i}(S_i)$ in both sides of the obtained inequality, we get $v_i(\cup_{j \in A \cup \{i\}} S_j) - \sum_{j \in A \cup \{i\}} v_j(S_j) > 0$, which contradicts Inequality (10). □

As a corollary, any a-approximation algorithm, together with access to demand queries, can be used to compute a significant $\mathcal{E}$-endowment CBE that has an a-approximation to the optimal social welfare.

7.2 A negative result for a set of *non-significant* endowment effects

In this section we show that there are endowment environments (that are not significant) for which Corollary 7.6 does not apply. Specifically, we show that for any $\beta < 1$, and endowment environment $\mathcal{E}$ such that $g_i^X(X) \leq \beta \cdot v_i(X)$ for all i and $X \subseteq M$, there exists an instance where an $\mathcal{E}$-endowment CBE with optimal social welfare does not exist. This is true even for XOS valuations. The following proposition establishes upper bounds on the social welfare that can be guaranteed in an $\mathcal{E}$-endowment CBE, as a function of β. We say that an allocation S is *supported* in an endowment equilibrium if there exist prices p such that (S, p) is an endowment equilibrium.

Proposition 7.8. *Consider any $\beta < 1$, and let $\mathcal{E}$ be an endowment environment such that $g_i^X(X) \leq \beta \cdot v_i(X)$ for all i. For every $\varepsilon > 0$, it holds that*

1. *There exists an instance such that no allocation with welfare better than $\frac{2-(\beta+\varepsilon)}{3-2(\beta+\varepsilon)} OPT$ can be supported in an $\mathcal{E}$-endowment equilibrium.*
2. *There exists an instance with* subadditive *consumers such that no allocation with welfare better than $\frac{4(1+\beta+\varepsilon)}{5+3(\beta+\varepsilon)} OPT$ can be supported in an $\mathcal{E}$-endowment equilibrium.*

3. *There exists an instance with* XOS *consumers such that no allocation with welfare better than* $\frac{8(1+\beta+\varepsilon)}{9+7(\beta+\varepsilon)}OPT$ *can be supported in an* $\mathcal{E}$*-endowment equilibrium.*

Proof. For the first statement, consider two identical items $\{s,t\}$, and two consumers. Consumer 1 has value 1 for a single item, and value x for two items. consumer 2 has value x for any non-empty set. In an optimal allocation each consumer gets a single item, with social welfare $1+x$. Let (p_1,p_2) be the consumers' prices. Suppose w.l.o.g. that in the optimal allocation consumer 1 receives s and consumer 2 receives t. By Definition 3.1, for consumer 1 to accept price p_1, it must hold that $p_1 \leq 1+g_1^{\{s\}}(\{s\})$. For consumer 1 to not want to add the other item, it must hold that $1+g_1^{\{s\}}(\{s\})-p_1 \geq x+g_1^{\{s\}}(\{s\})-p_1-p_2$; that is, $p_2 \geq x-1$. Similarly, for consumer 2 to accept price p_2 it must hold that $p_2 \leq x+g_1^{\{t\}}(\{t\})$, and to not prefer buying s at price p_1, it must hold that $x+g_2^{\{t\}}(\{t\})-p_2 \geq x-p_1$. We can now write the following sequence of inequalities:

$$x+g_2^{\{t\}}(\{t\})-(x-1) \geq x+g_2^{\{t\}}(\{t\})-p_2 \geq x-p_1 \geq x-(1+g_1^{\{s\}}(\{s\})).$$

By rearranging it follows that the constraints are satisfied only if $g_1^{\{s\}}(\{s\})+g_2^{\{t\}}(\{t\}) \geq x-2$. It is given that $g_1^{\{s\}}(\{s\})+g_2^{\{t\}}(\{t\}) \leq \beta(1+x)$. Therefore, if $\beta(1+x) < x-2$, i.e., if $\beta < \frac{x-2}{x+1}$ then the optimal allocation cannot be supported in an $\mathcal{E}$-endowment equilibrium.

Set $x=\frac{2-\beta'}{1-\beta'}$. For any $0<\beta'<1$ we have that $x>2$ and therefore $0<\frac{x-2}{x+1}<1$ and the analysis above holds. Therefore, if $\beta<\beta'$ then the optimal allocation cannot be supported in an $\mathcal{E}$-endowment equilibrium, and the next best allocation gives a $\frac{x}{1+x}=\frac{2-\beta'}{3-2\beta'}$ approximation to the optimal welfare. The result follows by setting $\beta'=\beta+\epsilon$.

For the second and third statements, consider a setting with three identical items $\{s,t,w\}$ and three consumers. Consumer 1 has valuation $(1,1,1+m)$, consumer 2 has valuation (m,m,m), and consumer 3 has valuation (a,a,a). We are interested in the case $1 \geq m > a$. Note that consumers 2 and 3 are unit demand. In an optimal allocation each consumer gets one item, the optimal social welfare is $1+m+a$, and the second best allocation achieves social welfare of $1+m$ (say, by giving all items to consumer 1). Suppose w.l.o.g. that in the optimal allocation consumer 1 receives s, consumer 2 receives t, and consumer 3 receives w. Each consumer i has a price p_i for her item. By Definition 3.1, for consumer 1 to be utility maximizing, she must not want to buy the two other items for a price of p_2+p_3 for a marginal increase of m, i.e., $p_2+p_3 \geq m$. Consumer 3 must prefer buying over not buying, i.e. $p_3 \leq a+g_3^{\{w\}}(\{w\})$. Consumer 2 must prefer her item over w, i.e., $m+g_2^{\{t\}}(\{t\})-p_2 \geq m-p_3 \Rightarrow g_2^{\{t\}}(\{t\}) \geq p_2-p_3$. Therefore, we have the following sequence of inequalities:

$$m \leq p_2+p_3 \leq p_3+p_3+g_2^{\{t\}}(\{t\}) \leq g_2^{\{t\}}(\{t\})+2(a+g_3^{\{w\}}(\{w\})) \leq \beta\cdot m+2(\beta\cdot a+a),$$

where the last inequality follows from by the assumption that $g_i^{S_i}(S_i) \leq \beta\cdot v_i(S_i)$ for all i. Rearranging, it follows that $\beta \geq \frac{m-2a}{m+2a}$, therefore, if $\beta < \frac{m-2a}{m+2a}$ then the optimal allocation cannot be supported in an $\mathcal{E}$-endowment equilibrium.

Set $a=\frac{m(1-\beta')}{2(1+\beta')}$, and conclude that if $\beta<\beta'<1$ then the optimal allocation cannot be an allocation of an $\mathcal{E}$-endowment equilibrium., and the next best allocation gives a $2/(2+\frac{m(1-\beta')}{2(1+\beta')})$ approximation to the optimal welfare.

For $m=1$, consumer 1 is subadditive, and it follows that if $\beta<\beta'$, then the next best allocation is a $\frac{4(1+\beta')}{5+3\beta'}$ approximation to the optimal social welfare.

For $m=1/2$, consumer 1 is XOS, thus if $\beta<\beta'$, then the next best allocation is a $\frac{8(1+\beta')}{9+7\beta'}$ approximation to the optimal social welfare. The results follow by setting $\beta'=\beta+\epsilon$. □

In particular, for $\beta \to 1$, the above shows that for XOS valuations, if $g_i^{S_i}(S_i) \leq (1-\varepsilon) \cdot v_i(S_i)$, then there is no $\mathcal{E}$-endowment equilibrium with optimal social welfare.

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A Appendix

A.1 Missing lemmas and propositions.

Lemma A.1. *Equation (3) holds if and only if $v^X(Y) = v(Y) + g^X(X \cap Y)$ for some $g^X : 2^X \to \mathbb{R}$.*

Proof. By definition of marginal valuation it holds that

$$v^X(Y \setminus X | X \cap Y) = v(Y \setminus X | X \cap Y)$$

if and only if

$$v^X(Y) - v(Y) = v^X(Y \cap X) - v(Y \cap X).$$

Letting $g^X(Y \cap X) \equiv v^X(Y \cap X) - v(Y \cap X)$ completes the proof. □

Lemma A.2. *Any endowment effect $\mathcal{E}$ satisfies the loss aversion inequality (Inequality (2)) if and only if every $g^X \in \mathcal{E}$ is weakly monotone, i.e., $g^X(Z) \leq g^X(X)$ for all $Z \subseteq X$.*

Proof. For any $X, Y \subseteq M$ it holds that

$$\begin{aligned} v^{X \cup Y}(X \cup Y) - v^{X \cup Y}(Y) &\geq v^Y(X \cup Y) - v^Y(Y) \iff \\ v(X \cup Y) + g^{X \cup Y}(X \cup Y) - (v(Y) + g^{X \cup Y}(Y)) &\geq v(X \cup Y) + g^Y(Y) - (v(Y) + g^Y(Y)) \iff \\ g^{X \cup Y}(X \cup Y) - g^{X \cup Y}(Y) &\geq g^Y(Y) - g^Y(Y) = 0 \end{aligned}$$

Note that the last inequality is equivalent to weak monotonicity of $g^{X \cup Y}$. □

Proposition A.3. *For any instance $(v_1, \ldots, v_n)$, a conditional equilibrium (S, p) is also an $\mathcal{E}^{AON}$-endowment equilibrium.*

Proof. Observe that for $g^{S_i} \in \mathcal{E}^{AON}$ and for all $Z \subsetneq S_i$ it holds that $g^{S_i}(Z) - p(Z) = -p(Z) \leq 0 \leq v_i(S_i) - p(S_i) = g^{S_i}(S_i) - p(S_i)$ where the second inequality follows by individual rationality in a conditional equilibrium. Proposition 4.7 completes the proof. □

A.2 Additional propositions

Corollary 5.2 and Theorem 4.1 imply that for XOS valuations, $\mathcal{E}^{AL} \not\prec \mathcal{E}^I$. The following is a slight strengthening:

Proof of Proposition 1.1. Fix $X \subseteq M$, need to show that for all $Z \subseteq X$, $g_I^X(Z|X \setminus Z) \leq g_{AL}^X(Z|X \setminus Z)$.

$$g_I^X(Z|X \setminus Z) = g_I^X(X) - g_I^X(X \setminus Z) = v(X) - v(X \setminus Z) \leq v(Z)$$

Where the last inequality follows by subadditivity. On the other hand

$$g_{AL}^X(Z|X \setminus Z) = g_{AL}^X(X) - g_{AL}^X(X \setminus Z) = v(X) - (v(X) - v(X \setminus (X \setminus Z)) = v(Z),$$

as required. □

The following proposition shows that $\mathcal{E}^{AON}$ is the strongest endowment effect in some class of endowment effects.

Proposition A.4. *For every valuation v, and every endowment effect $\mathcal{E}$ so that for all $X \subseteq M$, it holds that $g^X(X) \leq v(X)$ and $g^X \geq 0$, then $\mathcal{E} \prec \mathcal{E}^{AON}$.*

Proof. Fix a set $X \subseteq M$, and let $g^X \in \mathcal{E}$ and $g^X_{AON} \in \mathcal{E}^{AON}$. For any $Z \subsetneq X$, it holds that $g^X_{AON}(Z|X \setminus Z) = g^X_{AON}(X) - g^X_{AON}(X \setminus Z) = g^X_{AON}(X) = v(X)$. On the other hand, $g^X(Z|X \setminus Z) = g^X(X) - g^X(X \setminus Z) \leq g^X(X) \leq v(X)$ where the first inequality follows by $g^X \geq 0$, and the second is given in the proposition. □

Proposition A.5. *Let $\mathcal{E}^{PROP} = \{g^X(Z) = |Z| \cdot v(X) : X \subseteq M\}$. For any instance $(v_1, \ldots, v_n)$, there exists an $\mathcal{E}^{PROP}$-endowment equilibrium.*

Proof. Let consumer i be the consumer that maximizes the value of the grand bundle M. Consider the allocation of giving the grand bundle M to the consumer i, together with price $v_i(M)$ for each item.

Let us see that this pair of allocation and prices is an $\mathcal{E}^{PROP}$-endowment equilibrium.

The utility of consumer i is $g_i^M(M) + v_i(M) - m \cdot v_i(M) = v_i(M)$. Moreover, for any set $X \subsetneq M$, the utility of consumer i is

$$v_i^M(X) - |X| \cdot v_i(M) = v_i(X) \leq v_i(M),$$

therefore, consumer i does not wish to deviate. For any other consumer j, the utility from $X \subseteq M$ is

$$v_j^{\emptyset}(X) - |X| \cdot v_i(M) = v_j(X) - |X| \cdot v_i(M) \leq 0.$$

□