**Chapter 8.
The Logical Connective**

**'IF..., THEN...' (Implication)**

**If Alice peeks behind the curtain**

**then she will find the door that leads to the garden.**

**8.1 Worksheet 8a: Lead-in to the Logical Connective 'IF..., THEN...'**

You may begin instruction of this chapter with Worksheet 8a, to be performed as individual practice by each student. Notify the students upon completion of the task that they will be returning to this worksheet at the end of the chapter (Worksheet 8f); at that time they will be able to compare their initial understanding with that which they acquired during the course of the chapter. This will allow them to reflect upon the changes in their understanding along with the points where further clarification is still required (Worksheet 8g). For this reason it is recommended that you proceed with the remainder of the chapter upon completion of the worksheet, holding off discussion of this worksheet until the end of the chapter.

Because the lead-in worksheet is repeated again as a summary exercise, the sheet (together with proposed solutions) appears only once -- at the end of this chapter.

**8.2 A Look Inside the Text**

Read the two paragraphs from pages 7 and 8 of *Alice's Adventures in Wonderland* highlighted below. This excerpt will serve as a basis for a scientific discussion of the telescope as well as a discussion of the mathematical concept of "proportion".

Suddenly she came upon a little three-legged table, all made of solid glass; there was nothing on it but a tiny golden key, and Alice's first idea was that this might belong to one of the doors of the hall; but, alas! either the locks were too large, or the key was too small, but at any rate it would not open any of them. However, on the second time round, she came upon a low curtain she had not noticed before, and behind it was a little door about fifteen inches high: she tried the little golden key in the lock, and to her great delight it fitted!

**Alice opened the door and found that it led into a small passage, not much larger than a rat-hole: she knelt down and looked along the passage into the loveliest garden you ever saw. How she longed to get out of that dark hall, and wander about among those beds of bright flowers and those cool fountains, but she could not even get her head through the doorway; and even if my head would go through," thought poor Alice, "it would be of very little use without my shoulders. Oh, how I wish I could shut up like a telescope! I think I could, if I only knew how to begin." For, you see, so many out-of-the-way things had happened lately, that Alice had begun to think that very few things indeed were really impossible.**

**There seemed to be no use in waiting by the little door, so she went back to the table, half hoping she might find another key on it, or at any rate a book of rules for shutting people up like telescopes: this time she found a little bottle on it ("which certainly was not here before," said Alice,) and tied round the neck of the bottle was a paper label, with the words "DRINK ME" beautifully printed on it in large letters.**

It was all very well to say "Drink me," but the wise little Alice was not going to do *that* in a hurry. "No, I'll look first," she said, "and see whether it's marked '*poison*' or not;" for she had read several nice little stories about children who had got burnt,

**8.3 Scientific Discussion: The Telescope**

To expand the students' knowledge, it is advised to take the time for a closer look at this excerpt and hold a discussion with the students on various scientific aspects of the telescope and how it operates.

The telescope is an instrument used to observe far-away objects, such as stars. It allows visualization of details that cannot be seen by the naked eye.

The students may be encouraged to search for additional information about the telescope and to address questions such as those listed:

* When was the telescope invented and by whom?
* What kinds of telescopes exist today?
* How do the different telescopes work?
* Who uses telescopes?

**8.4 Discussion: The Mathematical Concept of "Proportion"**

The excerpt above may serve as a basis for discussion of the meaning of "proportion" and "proportional expansion (or reduction)".

Proportion in mathematics is the equivalence between numerical ratios *a*:*b=c:d* where *a, b, c, d* are real numbers, and *b, d* are non-zero.

Examples of mathematical problems that require considerations of proportion to solve may be found in textbooks on geometry (similar triangles), algebra and trigonometry.

**8.5 A Look Inside the Text**

Read the two paragraphs from pages 8 and 9 of *Alice's Adventures in Wonderland* highlighted below. These lines will serve as the basis for discussion of the logical connective 'IF ..., THEN ...'.

There seemed to be no use in waiting by the little door, so she went back to the table, half hoping she might find another key on it, or at any rate a book of rules for shutting people up like telescopes: this time she found a little bottle on it ("which certainly was not here before," said Alice,) and tied round the neck of the bottle was a paper label, with the words "DRINK ME" beautifully printed on it in large letters.

**It was all very well to say "Drink me," but the wise little Alice was not going to do *that* in a hurry. "No, I'll look first," she said, "and see whether it's marked '*poison*' or not;" for she had read several nice little stories about children who had got burnt, and eaten up by wild beasts, and other unpleasant things, all because they *would* not remember the simple rules their friends had taught them: such as, that a red-hot poker will burn you if you hold it too long; and that, if you cut your finger *very* deeply with a knife, it usually bleeds; and she had never forgotten that, if you drink much from a bottle marked "poison," it is almost certain to disagree with you, sooner or later.**

**However, this bottle was *not* marked "poison," so Alice ventured to taste it, and finding it very nice (it had, in fact, a sort of mixed flavour of cherry-tart, custard, pineapple, roast turkey, coffee, and hot buttered toast,) she very soon finished it off.**

"What a curious feeling!" said Alice. "I must be shutting up like a telescope."

And so it was indeed: she was now only ten inches high, and her face brightened up at the thought that she was now the right size for going through that little door into that lovely garden. First, however, she waited for a few minutes to see if she was going to shrink any further: she felt a little nervous about this: "for it might end, you know," said Alice to herself, "in my going out altogether, like a candle. I wonder what I should be like then?" And she tried to fancy what the flame of a candle looks like after the candle is blown out, for she could not remember ever having seen such a thing.

After a while, finding that nothing more happened, she decided on going into the garden at once; but, alas for poor Alice! when she got to the door, she found she had forgotten the little golden key, and when she went back to the table for it, she found she could not possibly reach it: she could see it quite plainly through the glass, and she tried her best to climb up one of the legs of the table, but it was too slippery; and when she had tired herself out with trying, the poor little thing sat down and cried.

**8.6 Worksheet 8b: The Logical Connective 'IF ..., THEN ...'**

The central topic of this chapter – the logical connective 'IF ..., THEN ...' – kicks off with Worksheet 8b, which begins with the excerpt shown above (Section 8.5).

**Remarks**:

* Worksheet 8b is to be performed in pairs or in small groups to allow for consultation.
* Upon completion of the worksheet, the students may present their answers to the class; a classroom discussion of their solutions is recommended.At this point it is advised that judgment of the answers be avoided; points of disagreement and misperceptions, however, should be noted.
* After presentation of the logical and mathematical foundations (Section 8.7 below), these answers should be revisited and discussed once again.

**Worksheet 8b and Proposed Solutions**

1. Find as many sentences as possible in the text that can be reworded such that they fit the format 'IF ..., THEN ...',and reword them as such.

1. If you hold a red-hot poker too long, then it will burn you.
2. If you cut your finger *very* deeply with a knife, then it usually bleeds.
3. If you drink much from a bottle marked "poison", then it is almost certain to disagree with you, sooner or later.

2. Can one tell with certainty that the bottle Alice drank from did not contain poison? Why?

It cannot be determined with certainty whether the bottle Alice drank from did not contain poison. There may be poison in the bottle even if it is not marked "poison". If the bottle were marked "poison" (assuming that the label is reliable), then we would know with certainty that it contained poison. But if the bottle is unmarked, we cannot determine what is in it.

3. How could Alice know with certainty that the bottle did not contain poison? Why?

Only by drinking the content of the bottle and examining its effect (don't try this at home!) could Alice know with certainty whether or not the bottle contained poison. Ther is no other way to verify this without laboratory access or some other means of verification.

**8.7 Logical and Mathematical Foundations**

Upon completion of Worksheet 8b, and after discussion of the students' answers, the following concepts and topics are to be presented:

1. Conditional statements
2. How to negate a conditional statement
3. What can be inferred from a conditional statement
4. What cannot be inferred from a conditional statement
5. The truth table for the implication connective
6. Rules of inference
7. Logical fallacies in drawing conclusions
8. Bi-directional conditional statements

**A. Conditional Statements**

Given two statements, two (typically different) conditional statements may be constructed from them by using the logical connective 'IF ..., THEN ...', as follows: Write the word IF followed by one of the statements, then add a comma and the word THEN followed by the second statement.

For example:

* Statement 1: It is raining in Wonderland.

Statement 2: The White Rabbit is wet.

One of the two conditional statements that may be obtained is "If it is raining in Wonderland, then the White Rabbit is wet." The first portion -- the protasis of the conditional statement -- which appears after the word IF ("it is raining in Wonderland"), is called the **antecedent**. The second portion -- the apodosis -- which appears after the word THEN, is called the **consequent**. The connective 'IF ..., THEN ...' is called the **implication connective.** The second conditional statement that can be constructed from these two is the reverse of the previous one: "If the White Rabbit is wet, then it is raining in Wonderland."

**Notes:**

* The conditional statement that can be derived may be meaningless or illogical. For example, the statement "If I buy shoes, then I will paint my car red" is not likely to be encountered in reality. Despite this, everything stated below about conditional statements applies to conditional statements of that nature as well.
* Even if we construct a conditional statement from two true mathematical statements, the result may be meaningless if there is no connection between the two components. An example of this is the statement "If two is a prime number, then all angles in a rectangle are 90°." Despite this, everything stated below about conditional statements applies to conditional statements of that nature as well.
* The meaning of a conditional statement relates, for the most part, to the existence of the consequent that follows from the assumption indicated in the antecedent. Yet it is important to keep in mind that logic deals with the structure of statements and the ramifications therein, and not with their content.
* If we denote the antecedent by *p* and the consequent by *q*, then the conditional statement "if *p*, then *q*" is denoted as follows:
*p**q*. The statement can also be expressed in other ways, such as "*p* implies *q*" and "*q* follows from *p*".
* As indicated above, the conditional statement *pq* has two parts -- the antecedent (*p*) and the consequent (*q*) -- each of which is a statement in itself. The two statements that compose the conditional statement could themselves be statements that consist of connectives or quantifiers (the OR here refers, of course, to Inclusive OR). Thus, for example, the connective NOT may appear on either part. As a result we have four possible cases:
1. *p* and *q* do not include negation ("If it is raining in Wonderland, then the White Rabbit is wet").
2. *p* includes negation and *q* does not ("If Alice doesn't find a key, then she will remain in the hall").
3. *p* does not include negation and *q* does ("If cats eat bats, then bats don't eat cats").
4. *p* and *q* both include negation ("If Alice hadn't seen the White Rabbit, then she wouldn't have entered the tunnel").
* Since the connective 'IF ..., THEN ...' is called the implication connective, such statements are sometimes called implication statements.

Everything stated below regarding conditional statements applies to each of the forms described above.

**B. How to Negate a Conditional Statement**

Let us examine two statements:

*p*: It is raining in Wonderland.

*q*: The White Rabbit is wet.

We assume that *p**q* holds true. In other words, the statement "If it is raining in Wonderland, then the White Rabbit is wet" is a true statement; that is, its content is true. Clearly its negation, therefore, is false.

But what is the negation of this statement?

Since we assumed that the conditional statement is true, obviously it could not be that it is raining in Wonderland and the White Rabbit is **not** wet. The negation of the conditional statement is therefore the false statement "It is raining in Wonderland **and** the White Rabbit is not wet." Note how the connective AND made its way into the conditional statement.

In general, negation of a conditional statement of the form "IF *p*, THEN *q*" is "*p* AND NOT *q*" or, in symbolic notation: *p**q.*

**C. What can be Inferred from a Conditional Statement**

In this section we present four different methods for expressing a conditional statement using equivalent statements; that is, statements which all follow from the conditional statement, and the conditional statement follows from each of them.

1. We found that the negation of the conditional statement *p**q* is *p**q.* It follows from this that the conditional statement *p*  *q* is equivalent to the statement: (*p**q*) OR ( *q**~*p ) OR ( ~*q**~p* ) (because there are only four possibilities). This may be written as:

*p**q*  (*p**q*)  (*p**q*) (*p**q*)

Thus, for example, from the statement "It is raining in Wonderland, then the White Rabbit is wet" the following statement can be inferred: "It is raining in Wonderland and the White Rabbit is wet, or it is not raining in Wonderland and the White Rabbit is wet, or it is not raining in Wonderland and the White Rabbit is not wet." The only case that cannot hold is that "It is raining in Wonderland and the White Rabbit has remained dry." Note that the conditional statement says nothing about the Rabbit's condition in the event that it is not raining.

1. Since the negation of the conditional statement *p**q* is: *p**q*, it follows that *p**q* is **equivalent** to the negation of its negation; that is, it is equivalent to (*p* *q*):

*p**q*  (*p**q*)

So, for example, from the statement "If it is raining in Wonderland, then the White Rabbit is wet", it can be inferred that "It cannot be the case that it is raining in Wonderland and the White Rabbit is not wet."

The reverse is true as well -- from the statement "It cannot be the case that it is raining in Wonderland and the White Rabbit is not wet", it can be inferred that "If it is raining in Wonderland, then the White Rabbit is wet."

1. In the discussion of De Morgan's Laws in Chapter 4, we found the equivalence:

(*p**q*)  *p**q*

In section C2 above we found that:

*p**q*  (*p**q*)

It follows from this (due to the transitive nature of equivalence) that:

*p**q*   *p**q*

In other words, the conditional statement *p**q* and the statement *p**q* are **equivalent** statements as well. So, for example, from the statement "If it is raining in Wonderland, then the White Rabbit is wet", it can be inferred that "It is not raining in Wonderland or the White Rabbit is wet." The reverse is true as well -- from the statement "It is not raining in Wonderland or the White Rabbit is wet" it can be inferred that "If it is raining in Wonderland, then the White Rabbit is wet."

1. We stated that if it is raining in Wonderland, then the White Rabbit is wet. Could it be that the White Rabbit is not wet, and it is raining in Wonderland? -- No. We saw in Section B above that this could not be the case. In other words, if the White Rabbit is not wet, then it is undoubtable that it is not raining in Wonderland. This last statement is a conditional statement as well. It is different from the original statement, but expresses the identical content. The difference between them is that the antecedent and the consequent have reversed roles and in parallel have taking on a negation (that is, their truth values have flipped). In general, the new statement *q**p*is **equivalent** to the original *p**q*. This may be expressed alternatively as:

*p**q*  *q**p*

These are referred to as **contrapositives** of one another.

**D. What Cannot be Inferred from a Conditional Statement**

Once again let us examine the statement "If it is raining in Wonderland, then the White Rabbit is wet." We saw earlier than this statement does not indicate anything about the White Rabbit when it is not raining in Wonderland. Note in particular that the conditional statement "If it is **not** raining in Wonderland, then the White Rabbit is **not** wet" **does not** follow from the given statement. The White Rabbit could be wet even it is is not raining in Wonderland (it may have gotten wet from a water sprinkler, for example). In general, it is an error (and a common one) to infer the **inverse**, that is, *pq*, from the statement *p**q*.

Similarly, it is an error (again, a common one) to infer the **converse**, *q**p,* from the statement *p**q.* The statement "If it is raining in Wonderland, then the White Rabbit is wet" does not indicate that "If the White Rabbit is wet, then it is raining in Wonderland." As we indicated, the White Rabbit could have gotten wet from a water sprinkler.

The tendency to erroneously infer from the statement *p**q* one of the statements: *p*~*q* or *q**p*, apparently stems from a sense of verbal symmetry or from imprecise use of "IF ..., THEN ..." in daily language. An example of this is the use of "IF ..., THEN ..." as a promise or a threat: A child who is promised "If you finish your dinner you will get dessert," understands this as a threat -- if he does not finish his dinner, he will not get dessert. This interpretation encourages him to finish his dinner, but in truth, the threat does not follow from the promise made to him, although it is true that that was the intent. According to the promise made, he may get dessert even if he does not finish his dinner. A child who is told "If it rains I will pick you up from school", may justifiably ask: "And what if it doesn't rain?" The answer does not follow from the promise, although it may be assumed that what is meant is "ONLY if it rains will I pick you up from school; and if it doesn't rain, I will not."

**E. The Truth Table for the Implication Connective**

Let us construct a truth table for the two statements *p* and *q*, and for the compound statement *p**q*, obtained by using the logical connective 'IF ..., THEN ...'. Recall that there are four combinations of truth values for the statements *p* and *q*:

|  |  |
| --- | --- |
| ***q*** | ***p*** |
|  *T* |  *T* |
| *F* |  *T* |
|  *T* | *F* |
| *F* | *F* |

In Section C above we found that the conditional statement *p**q* is equivalent to each of the following three compound statements:

 (*p**q*)(*p**q*)(*p**q*)

(*p**q*)

*p**q*

We construct the truth table for the latter equivalent statement, using our knowledge of the NOT and AND connectives from Chapters 1 and 2.

|  |  |  |  |
| --- | --- | --- | --- |
| ***p**q*** | ***p*** | ***q*** | ***p*** |
|  *T* | *F* |  *T* |  *T* |
| *F* | *F* | *F* |  *T* |
|  *T* |  *T* |  *T* | *F* |
|  *T* |  *T* | *F* | *F* |

But, as indicated:

*p**q*  *p**q*

and therefore the truth table for the implication connective is:

|  |  |  |
| --- | --- | --- |
| ***p*  *q*** | ***q*** | ***p*** |
|  *T* |  *T* |  *T* |
| *F* | *F* |  *T* |
|  *T* |  *T* | *F* |
|  *T* | *F* | *F* |

**Remark:**

* Students may be called upon to prepare a truth table for one of the first two equivalent statements. This should be performed based on the knowledge they acquired on the connectives OR, AND and NOT in Chapters 1 through 3. The results, of course, are identical.

**F. Rules of Inference**

The rules of inference are a template for determining that a particular conclusion necessarily follows from a given set of data. There are two basic rules of inference:

Inference Rule #1: Given the conditional statement "IF *p*, THEN *q*"; that is, that *p**q* is a true statement, and given *p*; that is, that the antecedent of the conditional statement is also a true statement, it follows that the consequent statement *q* is necessarily true as well.

For example:

* Given: 1. If it is raining in Wonderland, then the White Rabbit is wet.

 2. It is raining in Wonderland.

 Conclusion: The White Rabbit is wet.

This rule of inference is called **modus ponendo ponens**, or **modus ponens** for short. This is the most basis rule of inference.

It is written in statement notation as:



 *p*

\_\_\_\_\_\_\_\_\_\_

 *q*

**Note:**

* This rule is also known as the **Law of Detachment**.

Inference Rule #2: Given the conditional statement "If *p*, then *q*"; that is, that *p**q* is a true statement, and given ~*q*; that is, that the consequent of the conditional statement is false, then it follows necessarily that *p*. That is, the antecedent statement *p* is false as well.

For example:

* Given: 1. If it is raining in Wonderland, then the White Rabbit is wet.

 2. The White Rabbit is not wet.

Conclusion: It is not raining in Wonderland.

This inference rule is known as **modus tollendo tollens**, or **modus tollens**, for short.

It is written in statement notation as:



*q*

 \_\_\_\_\_\_\_\_\_\_

 *p*

**Notes:**

* We saw in Section 4C that the conditional statement "IF *p*, THEN *q*" is equivalent to its contrapositive "IF *q*, THEN *p*". Application of modus tollens, therefore, is equivalent to application of modus ponens.

It is written in statement notation as:

*q*  *p*

 *q*

\_\_\_\_\_\_\_\_\_

 *p*

* As mentioned above, it is important to distinguish between a causal relationship that may be expressed by the content of the conditional statement, and its structure. Thus, it should be noted that the rules of inference determine an outcome based on the structure of the given statements and not by their content.

For example:

* Whoever assumes that the statements given below are true:

- If ears of corn grow in Wonderland, then the White Rabbit will buy a new watch.

- Ears of corn grow in Wonderland.

must draw the conclusion that the White Rabbit will buy a new watch -- even though the conditional statement has a causal relationship that is unclear to us.

**G. Logical Fallacies in Drawing Conclusions**

As mentioned above, there is a natural tendency to infer from the conditional statement "IF *p*, THEN *q*" that "IF NOT *p*, THEN NOT *q*." This fallacy may be expressed as:



 ~*p*

\_\_\_\_\_\_\_\_\_\_

 *q*

So, for example, assuming that the statement "If it is raining in Wonderland, then the White Rabbit is wet" is true, and someone in Wonderland is looking out of the window and sees that it is not raining, does this mean that the White Rabbit is not wet? This question cannot be answered conclusively. The given information does not suffice to answer the question. The White Rabbit may be wet and it may not be wet. What we do know is what happens to the White Rabbit when it **is** raining in Wonderland. The conditional statement does not tell us anything about what happens to the White Rabbit when it is **not** raining in Wonderland.



 *p*

\_\_\_\_\_\_\_\_\_\_

There is insufficient information to draw conclusions one way or the other.

Such a logical fallacy is known as **denying the antecedent**.

An additional kind of logical fallacy is **accepting the consequent**.

Let us assume that we met the White Rabbit and saw that it is wet. Does this mean that it is raining in Wonderland?

Clearly it cannot be determined conclusively whether the White Rabbit is wet from the rain, from a water sprinkler or that it just came out of the shower. Therefore no conclusion can be drawn regarding rain in Wonderland.

This fallacy may be written as follows:



 *q*

\_\_\_\_\_\_\_\_\_\_

 *p*

The truth is that --



 *q*

\_\_\_\_\_\_\_\_\_\_

 There is insufficient information to draw any conclusions.

**H. Bi-directional Conditional Statements**

*p, q* are two statements. Two conditional statements may be constructed from them:  and . We say that each conditional statement is the **converse** of the other. We saw above that one of the conditional statements may be true, while the other doesn't necessarily follow from it; that is, perhaps it is true, and perhaps it is false. In other words, all possibilities are open -- both could be false statements, both could be true statements or one could be true and the other false. In the case that both antecedents are true; that is,  and  both hold, then we also say *p* if and only if *q*,or *q* if and only if *p.* This is notated in short as:
*p* if and only if *q,* and is denoted *.*

Examples:

* *p*: A given quadrilateral is a parallelogram.

*q*: The diagonals of this quadrilateral bisect each other.

The resultant statements are as follows:

If a quadrilateral is a parallelogram (*p*), then its diagonals bisect each other (*q*).

If the diagonals of a quadrilateral bisect each other (*q*), then the quadrilateral is parallelogram (*p*).

Since both of the two conditional statements are true, we can state that a quadrilateral is a parallelogram if and only if its diagonals bisect each other, or that diagonals in a quadrilateral bisect each other if and only if the quadrilateral is a parallelogram.

* Two triangles are congruent if and only if the three sides of one are congruent to the corresponding three sides of the other respectively.
* The quadratic equation  in which , , has two solutions if and only if the discriminant .

**Notes:**

* Decomposition of the conditional statement into its respective components must be performed carefully. For example, in the last case presented above, the conditions that apply to  were excluded from the second half of the bi-directional conditional statement due to linguistic considerations. Upon decomposition, these need to be restated.
* What was presented in section C above with regard to statements equivalent to conditional statements can now be written as follows:

(*p*  *q*)  [(*p**q*)  (*p**q*)  (*p* *q*)]

(*p*  *q*)  [(*p* *q*)]

(*p*  *q*)  [*p**q*]

(*p*  *q*)  [*q* *p*]

**Remark: \*\*up to here\*\***

* Students may be called upon to try phrasing De Morgan's laws (Chapter 4) using bi-directional conditional statements.

**Note:**

* As indicated, in logic -- the language of mathematics -- the implication connective is denoted by convention using an arrow, and the bi-directional conditional statement is denoted using a double arrow. On the other hand, in mathematics these are conventionally indicated with the symbols ⇒ ,  rather than → , ↔ , respectively.

Examples:

* +  for each *a* that is a natural number;
	+ (*ABC*  *DEF*)  [(*AB* = *DE*)  (*BC* = *EF*)  (*CA* = *FD*)]

In Chapter 9 we discuss the meaning of "if and only if" once again, in the context of necessary and sufficient conditions.

**8.8 Review of Answers to Worksheet 8b**

After the students have become familiar with the logical foundations, it is advised that they go back and review their answers to Worksheet 8b and correct them if necessary.

**8.9 Worksheet 8c: The Logical Connective 'IF ..., THEN ...'– Drawing Conclusions using Rules of Inference**

Worksheet 8c addresses inference from statements containing the logical connective 'IF ..., THEN ...'.

**Remarks**:

* This worksheet may be performed as individual practice, in pairs or in small groups, with group discussion encouraged.
* Upon completion of the worksheet, the students may present their answers to the class; a classroom discussion of these solutions is recommended.

**Worksheet 8c and Proposed Solutions**

Each part of the table below contains two statements followed by a conclusion. Indicate the given statements and the proposed conclusion using statement notation. Then examine the proposed conclusion in light of the given statements, and determine whether the conclusion does in fact follow from them.

|  |  |  |
| --- | --- | --- |
| **PART** | **GIVEN** | **THE CONCLUSION** |
| 1 | 1. If you hold a red-hot poker too long, then it will burn you.
2. I got a burn on my finger.
 | I held a red-hot poker too long. |

Let us designate the statements as follows:

*p*: (People) hold red-hot pokers for too long.

*q*: Red-hot pokers burn.

We now indicate the given information in statement notation.

 

 *q*

\_\_\_\_\_\_\_\_\_\_

*p*

|  |  |  |
| --- | --- | --- |
| ***p*  *q*** | ***q*** | ***p*** |
|  *T* |  *T* |  *T* |
| *F* | *F* |  *T* |
|  *T* |  *T* | *F* |
|  *T* | *F* | *F* |

We construct a truth table corresponding to the logical connective 'IF ..., THEN ...'.

It is given that *p*  *q* is TRUE. This bit of information corresponds to the first, third and fourth rows of the truth table for the implication connective.

It is also given that *q* is a TRUE statement. This narrows the possibilities to the first and third rows.

|  |  |  |
| --- | --- | --- |
| ***p*  *q*** | ***q*** | ***p*** |
|  *T* |  *T* |  *T* |
| *F* | *F* |  *T* |
|  *T* |  *T* | *F* |
|  *T* | *F* | *F* |

From these two rows we see that statement *p* could be TRUE or FALSE; therefore it cannot be determined whether or not the proposed conclusion presented in the table ("I held a red-hot poker too long") is TRUE.

Or, phrased in words: Although it is possible that I got a burn because I held a red-hot poker too long, it is also possible that the burn was caused by something else. (A common logical fallacy would be that the conclusion is TRUE).

|  |  |  |
| --- | --- | --- |
| **PART** | **GIVEN** | **THE CONCLUSION** |
| 2 | 1. If you cut your finger *very* deeply with a knife, then it bleeds.
2. My finger is not bleeding.
 | I did not cut a *very* deep cut on my finger with a knife.  |

Let us designate the statements as follows:

*p*: (People) cut a *very* deep cut on their finger with a knife.

*q*: (Someone's) finger is bleeding.

We now indicate the given information in statement notation.



*q*

\_\_\_\_\_\_\_\_\_\_

*p*

We construct a truth table corresponding to the logical connective 'IF ..., THEN ...'.

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| ***p*  *q*** | ***q*** |  |  | ***p*** |
|  *T* |  *T* |  |  |  *T* |
| *F* | *F* |  |  |  *T* |
|  *T* |  *T* |  |  | *F* |
|  *T* | *F* |  |  | *F* |

It is given that the compound statement *p->q* is TRUE. For this reason, just as in the case in the first question, in accordance with the truth table for this connective, the first, third and fourth rows match the given information.

It is also given that *q* is FALSE. Thus only the fourth row matches the given information.

From this row we can see that statement *p* is FALSE; therefore, the proposed conclusion in the table is TRUE.

Or, phrased in words: According to the first bit of information, there can't be a *very* deep cut in the finger if the finger is not bleeding. Therefore the second piece of information, which establishes that the finger is not bleeding, invalidates the possibility that the finger was cut.

|  |  |  |
| --- | --- | --- |
| **PART** | **GIVEN** | **THE CONCLUSION** |
| 3 | A. If you drink from a bottle marked "poison", then it will disagree with you.B. Alice did not drink from a bottle marked "poison". | Nothing disagreed with Alice. |

Let us designate the statements as follows:

*p*: (People) drink from a bottle marked "poison".

*q*: Something disagreed with (people).

We now write the given information in statement notation:



*p*

\_\_\_\_\_\_\_\_\_\_

 *q*

We construct a truth table corresponding to the logical connective 'IF ..., THEN ...'.

|  |  |  |
| --- | --- | --- |
| ***p*  *q*** | ***q*** | ***p*** |
|  *T* |  *T* |  *T* |
| *F* | *F* |  *T* |
|  *T* |  *T* | *F* |
|  *T* | *F* | *F* |

It is given that the compound statement *p->q* is TRUE. For this reason, just as in the case in the first question, in accordance with the truth table for this connective, the first, third and fourth rows of the table match the given information.

It is also given that *p* is FALSE. Only the third and fourth rows match this information.

From these two rows we see that statement *q* can be either TRUE or FALSE; therefore it cannot be determined whether or not the proposed conclusion presented in the table ("Nothing disagreed with Alice") is TRUE.

Or, phrased in words: All we know is that nothing disagreed with Alice as a result of drinking from the bottle of poison, but this does not guarantee that there was not something that disagreed with her as a result of something else.

|  |  |  |
| --- | --- | --- |
| **PART** | **GIVEN** | **THE CONCLUSION** |
| 4 | 1. If the bottle is not marked "poison", then the liquid inside tastes like custard.
2. The bottle is not marked "poison".
 | The liquid in the bottle tastes like custard. |

Let us designate the statements as follows:

*p*: The bottle is not marked "poison".

 *q*: The liquid in the bottle tastes like custard.

We now indicate the given information in statement notation.



 *p*

\_\_\_\_\_\_\_\_\_\_

 *q*

We construct a truth table corresponding to the logical connective 'IF ..., THEN ...'.

|  |  |  |
| --- | --- | --- |
| ***p → q*** | ***q*** | ***p*** |
| *T* | *T* | *T* |
| *F* | *F* | *T* |
| *T* | *T* | *F* |
| *T* | *F* | *F* |

It is given that the compound statement *p->q* is TRUE. For this reason, just as in the case in the first question, in accordance with the truth table for this connective, the first, third and fourth rows of the table match the given information.

It is also given that *p* is a TRUE statement. Only the first and second rows match this information.

Therefore only the first row matches both bits of information.

From this row we obtain that statement *q* must be TRUE; therefore the proposed conclusion is TRUE.

Or, phrased in words: The conclusion is true by applying the first rule of inference (*modus ponens*).

**8.10 Worksheet 8d: The Logical Connective 'IF ..., THEN ...'– The Four-Card Problem**

The problem included in this worksheet is known as "the four-card problem" or the "Wason selection task". The problem (unrelated to Alice, of course), was first presented in 1972 by two psychologists: Peter Wason and Philip Johnson-Laird.[[1]](#footnote-1)

**Worksheet 8d and Proposed Solutions**

1. Alice and the White Rabbit like to ask each other riddles. One day Alice laid four cards on the table, with only one side of them visible.

|  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- |
| **A** |  | **D** |  | **4** |  | **7** |

Alice told the White Rabbit that on one face of each card is a letter, and on the other face is a number.

Then Alice posed the following question to the White Rabbit:

"Which card or cards must be turned over in order to prove or disprove the following proposition:

For each card, if a card shows a vowel on one face, then its opposite face has an even number."

What would you answer if you were the White Rabbit?

In 1972, when the two psychologists Wason and Johnson-Laird presented the problem to 128 students, 59 of them (46%) claimed that the cards showing A or 4 must be turned over. Forty-two students (33%) claimed that the card showing A must be turned over. Only 5% of the students gave the right answer.

The most common reasoning given for choosing the card showing A was: If on the opposite side of the card is an even number, then the assertion is validated at least partially. On the other hand, if on the opposite face is an odd number, then the assertion has been refuted.

The reasoning behind this explanation is: A is a vowel. Let us assume that the assertion is true. If so, on the opposite side of the card there must be an even number. If we flip the card and find an odd number, then the assertion is invalid. This thinking leads to proof by negation, and it is in fact necessary in this case to turn this card over.

The most common explanation for selecting the card showing 4 was: We will check whether there is a vowel on the other side. This answer comes from a logical fallacy. Let us assume that we turned over the card showing 4 and did not find a vowel on the opposite face. Have we refuted the assertion?

Obviously if we don't find a vowel on the back, we have not yet refuted the assertion. The assertion does not refer at all to the case where there is a consonant on the card, nor does it determine that **only** vowels have even numbers on the back. Therefore there is no point in selecting this card.

The other card that must be turned over in order to examine the truth of the assertion is the card showing the number 7. For the assertion to be true, a vowel may not appear on the back of the card, since 7 is an odd number. If a vowel appears on the back of the card, then the assertion has been refuted. Here too, the proof is by negation.

Would there be a point in turning over the card showing D? - Let us assume that the card showing D was turned over, and on the back was an even number. Has the assertion been proven by this? Let us assume that on the other side was an odd number. Has the assertion been refuted by this? The answer to both of these questions is no, since the assertion does not relate to consonants. Therefore there is no point in turning this card over.

To summarize -- the cards that must be turned over are the ones labeled A and 7. It is possible that the assertion will be refuted after turning over only one of them. But if the first card supports the assertion, then the second card must still be turned over, in order to see whether it, too, supports the assertion.

**Remark:**

* Wason and Johnson-Laird presented another version of the problem, in which they retained the same assertion, but presented only two cards -- on one there was a 1 and on the other a 2. You may wish to present this problem to the students after the discussion of Worksheet 8d.

**8.11 Worksheet 8e: The Logical Connective 'IF ..., THEN ...'– Negation of the Conditional Statement**

Worksheet 8e addresses the negation of the conditional statement.[[2]](#footnote-2)

**Remarks**:

* This worksheet may be performed as individual practice, in pairs or in small groups, with group discussion encouraged.
* Upon completion of the worksheet, the students may present their answers to the class; a classroom discussion of these solutions is recommended.

**Worksheet 8e and Proposed Solutions**

A. In each picture below there is a White Rabbit and a TV set. Examine each picture carefully and read the sentences that appear below them. Then, circle the numbers of the pictures that contradict the content of the statement. Explain your answer.

1

1

2

3

5

4

6

![j0356865[1]]()![j0356865[1]]()![j0356865[1]]()



If the TV is on, then the White Rabbit is not playing the trumpet.

Picture 3 contradicts the content of the statement. The conditional statement is: If the TV is on, then the White Rabbit is not playing the trumpet. Each picture that presents a TV set that is on alongside a Rabbit that **is** playing the trumpet contradicts this statement. This can be stated formally as follows:

*p*: The TV set is on. *q*: The White Rabbit is not playing the trumpet.

The given conditional statement:  is equivalent to , and therefore its contradiction is . Note that the cases  do not contradict the conditional statement; that is, the pictures in which the TV is off do not contradict the conditional statement, since the conditional statement does not provide any information about the behavior of the Rabbit when the TV is off. Alongside a TV that is turned off, there could be two possible cases -- either the Rabbit is playing the trumpet, or it is not.

B. In each picture below there is a White Rabbit and a TV set. Examine each picture carefully and read the sentences that appear below them. Then, circle the numbers of the pictures that **contradict** the content of the statement. Explain your answer.

1

1

2

3

**![j0356865[1]]()![j0356865[1]]()![j0356865[1]]()**

5

4

6

****

If the White Rabbit is not playing the trumpet, then the TV is on.

Pictures 5 and 6 contradict the content of the statement.

The conditional statement here is: If the White Rabbit is not playing the trumpet, then the TV is on. Note that this conditional statement is the reverse of the previous one (8.7h above). Every picture that presents the White Rabbit not playing the trumpet alongside a TV that is **not** on, contradicts this statement. This can be stated formally as follows:

*p*: The White Rabbit is not playing the trumpet.

*q*: The TV set is on.

The conditional statement:  is equivalent to ; therefore its contradiction is . Note that the cases  do not contradict the conditional statement; that is, the pictures in which the Rabbit is playing the trumpet do not contradict the conditional statement, since the conditional statement provides no information about the state of the TV when the White Rabbit **is** playing the trumpet.

**8.12 Worksheets 8f and 8g: Summary Exercise for Conditional Statements**

The summary exercise is composed of two worksheets: 8f and 8g.

**Remarks**:

* Worksheet 8f is to be performed as individual practice by the students.In Worksheet 8g (which appears in the students' workbook only) the students will compare their current answers with the answers they gave at the start of the chapter (Worksheet 8a).
* Upon completion of the summary exercise a classroom discussion is recommended to consider the changes that have taken place in the students' understanding and perceptions through the course of this chapter, as well as the particular difficulties encountered with the subject matter.

**Worksheet 8f and Proposed Solutions**

Below is a table with four items. Each item contains two statements followed by a proposed conclusion.

For each proposed solution, select one of the following three options. Explain your choice.

A. It follows from the statements that the conclusion is true.

B. It does not necessarily follow from the statements that the conclusion is true

C. It follows from the statements that the conclusion is false

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **THE GIVEN****STATEMENTS** | **PROPOSED CONCLUSIONS** | **IT FOLLOWS FROM THE STATEMENTS THAT THE CONCLUSION IS TRUE** | **IT DOES NOT NECESSARILY FOLLOW FROM THE STATEMENTS THAT THE CONCLUSION IS TRUE** | **IT FOLLOWS FROM THE STATEMENTS THAT THE CONCLUSION IS FALSE** |
| A | 1. If the White Rabbit comes to the party, then Alice will be happy.2. The White Rabbit did not come to the party. | Alice is happy. |  | It cannot be determined whether or not Alice is happy, since the conditional statement provides information about Alice only in the case that the Rabbit comes to the party.  |  |
| B | 1. If the White Rabbit comes to the party, then Alice will be happy.2. Alice is not happy. | The White Rabbit came to the party. |  |  | The given conditional statement assures that when the proposed conclusion is true, Alice is happy. But it is also given that Alice is not happy. Therefore it could not be that the proposed conclusion is true. |

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | **THE GIVEN****STATEMENTS** | **PROPOSED CONCLUSIONS** | **IT FOLLOWS FROM THE STATEMENTS THAT THE CONCLUSION IS TRUE** | **IT DOES NOT NECESSARILY FOLLOW FROM THE STATEMENTS THAT THE CONCLUSION IS TRUE** | **IT FOLLOWS FROM THE STATEMENTS THAT THE CONCLUSION IS FALSE** |
| C | 1. If the White Rabbit comes to the party, then Alice will be happy.2. The White Rabbit came to the party. | Alice is happy. | The given conditional statement assures that Alice will be happy if the White Rabbit comes to the party. When the White Rabbit does in fact come to the party, Alice is happy. |  |  |
| D | 1. If the White Rabbit comes to the party, then Alice will be happy.

2. Alice is happy. | The White Rabbit came to the party. |  | It cannot be determined whether the White Rabbit came to the party or not. Perhaps Alice is happy for other reasons. The conditional statement provides information about one of the reasons for Alice's happiness, but does not establish this as the only reason.  |  |

1. Additional details may be found in the book:

 [Wason, P. C. & Johnson-Laird, P. N. (1972). Psychology of reasoning: Structure and content. Harvard University Press.](http://www.amazon.com/exec/obidos/tg/detail/-/0674721276/qid%3D1124494637/sr%3D8-1/ref%3Dsr_8_xs_ap_i1_xgl14/103-7809492-0639840?v=glance&s=books&n=507846) [↑](#footnote-ref-1)
2. This worksheet was inspired by the following sources:

Hadar, N. (1975). *Children’s Conditional Reasoning.* Ph.D. dissertation. University of California, Berkeley. ERIC accession number ED118359, RIE June 1976.

Hadar, N. (1977). An intuitive approach to the logic of implication. *Educational Studies in Mathematics, 8*, 413-438. [↑](#footnote-ref-2)