OPTIMAL SELLER STRATEGY IN OVERLAPPING AUCTIONS

1. Introduction

The ~~past decade's~~ burgeoning of online auctions over the past decade has given rise to significant changes in the ways that products are sold. One important trend in online auctions is an increase in concurrent auctions, in which multiple sellers sell identical or similar products in multiple auctions on popular consumer-to-consumer (C2C) and seller business-to-consumer (B2C) online platforms (e.g., eBay, eBid, Bonanza, graysonline.com and Ubid) and single-seller B2C company websites (e.g., Dell auctions and Sam's Club auctions). (Please see Web Appendix A, Table WA1, for an overview of online auction websites.)~~.~~ Non-profit organizations are also embracing concurrent auctions such as government auctions and liquidation auctions (e.g., BStock, GovDeals, and VDC Canada), penny auctions (e.g., DealDash, OrangeBidz, and QuiBids), and charity auctions (e.g., CharityBuzz and Shopgoodwill).[1]

On single-seller websites (e.g., Dell, Sam's Club, and penny-auction websites), a single business, selling multiple items over time, needs to determine the best way to do so. In particular, the timing/degree of overlap between auctions for similar items has important implications for bidders, because the overlap influences the quality of the information bidders can obtain~~ed~~ from previous auctions, their forward-looking actions for~~of~~ future auctions, and their cross-bidding among overlapp~~ed~~ing auctions. For the seller, the degree of overlap is important, as the overlap determines the number of items that can be sold in a certain time period and influences the final bids in concurrent auctions. As such, the amount of time two auctions overlap~~ped~~ influences bidder strategy and seller revenue (Bapna et al., 2009). Data collected in July 2017 from Sam's Club illustrates that over half of the multiple concurrent auctions overlap to various degrees in different categories (see Table 1).

[1] Overlapping auctions are also prevalent in business-to-business (B2B) settings such as electricity auctions (Maurer and Barroso, 2011; Mazzi et al., 2015), utilities and spectrum auctions (Cramton, 2002; Goetzendorff et al., 2018), and transportation auctions for the sourcing of truckloads (Lindsey and Mahmassani, 2017; Xu, Xiu, and Huang, 2017).

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Sam's ~~c~~Club uses ascending bid (English) auctions, similar to what we model in this paper. Auctions generally sell one unit per auction. ~~-~~And in many instances, multiple auctions for the identical items run concurrently, with the same duration but different starting and ending times (i.e., auctions are partially overlapping). Most auction durations vary (from 12 hours to 3 days), and they ~~are~~ overlap~~ped~~ by various degrees. Bidders include regular ~~consumers~~users of Sam's C~~c~~lub~~,~~ as well as~~and~~ additional customers attracted to auctions.

On multiple-seller sites (e.g., eBay, eBid, charity auctions, and transportation auctions), sellers need to determine the optimal timing of their auctions relative to competing ones. To facilitate this calculation~~effort~~, some websites provide recommendations on the timing of auctions (e.g., eBay's Seller Guide[2]), based on competitive market conditions.

--- Insert Table 1 about here ---

Therefore, determining the optimal way to sell multiple identical or similar items over time~~—~~—in simultaneous, sequential, or partially overlapping auctions—for both single- and multiple-seller auction websites has important implications for seller revenue. However, this question has not been addressed in the literature. The objective of the current research is to study the impact of degree of overlap on auction revenue. In particular, we study the impact of bidder learning, rate of bidder entry and withdrawal, bidder forward-looking behavior, and time discounting on optimal auction format, and determine under what conditions it is optimal to use sequential, simultaneous or partial overlapping auctions.

Most previous research treats online concurrent auctions as individual independent auctions, ignoring the interdependency across competing auctions and its impact on bidder behavior, such as cross-bidding, learning, and forward-looking strategies. Bidders often switch (cross-bid) between auctions, which has been frequently observed in empirical research (Anwar et al., 2006; Haruvy and Popkowski Leszczyc,

[2] https://www.auctionnudge.com/guides/what-time-is-best-to-list-and-therefore-end-your-auctions/ (accessed April 3, 2018).

2010a; Haruvy, Popkowski Leszczyc, and Ma, 2014). Haruvy et al. (2014) report significant cross-bidding in simultaneous auctions (53.0% cross-bidding), sequential auctions (59.48% cross-bidding) and in partial overlapping auctions (15.67% cross-bidding).[3] This observation is also consistent with theoretical predictions of bidding in multiple auctions (Peters and Severinov, 2006). In addition, bidders may adapt their bidding strategies as a result of *learning* from past auction prices, which reduces bidders' uncertainty about the product value (Chang, 2014; Goes, Karuga, and Tripathi, 2010 and 2012; Kagel and Levin, 1986). There is considerable evidence of consumer learning in auctions (Wang and Hu, 2009; Pownall and Wolk, 2013; Pilehvar et al., 2017). Inexperienced bidders learn from their experience in previous auctions and learn from concurrent and recently completed auctions (Pilehvar et al., 2017). They may also adapt their bidding strategies as a result of *forward-looking* behavior; that is, they bid less after anticipating the occurrences and prices of future auctions (Jofre-Bonet and Pesendorfer, 2003; Zeithammer, 2006, 2007). Several papers have provided empirical support for bid shading, observing lower ending prices when similar items are auctioned close in time (Zeithammer, 2006; Bapna et al., 2009; Pownall and Wolk, 2013).

Researchers have recently begun exploring the interactions among online concurrent auctions, but they have mostly limited their explorations to simultaneous (full overlap) or sequential auctions (zero overlap), two extreme cases of concurrent auctions. In the following, we briefly discuss the three common formats of overlapping auctions.

1.1. Fully-overlapping (Simultaneous) Auctions

Cross-bidding (i.e., switching among auctions) is a major feature of *simultaneous* auctions, as rational bidders should bid incrementally and in the auction with the lowest current price (Peters and Severinov, 2006). On the one hand, cross-bidding may depress prices as bidders bid in the lowest-price auction (Anwar et al., 2006; Bapna, 2009). On the other hand, it results in more bidders and bids in both auctions,

[3] Most cross-bidders make a single switch to another concurrent auction, especially for partial overlapping auctions.

which increases the intensity of competition and ending prices (Beil and Wein, 2009).

1.2. Non-overlapping (Sequential) Auctions

Sequential auctions provide rational bidders the opportunity to look ahead as they anticipate the occurrence of future auctions, and also allow bidders to learn from the outcomes of earlier auctions.[4] Bidders' forward-looking behavior results in less ~~-~~aggressive bidding and thus lower prices (due to bid shading) in the current auction, as high ~~-~~valuation bidders trade off winning the current auction with the opportunity to win in a future auction at a lower price (Engelbrecht-Wiggans, 1994; Jofre-Bonet and Pesendorfer, 2003; Zeithammer, 2006, 2007).[5] ~~-~~Bidders' learning from preceding auctions, on the other hand, leads to more ~~-~~aggressive bidding (and thus higher prices) in future auctions, because learning reduces bidders' uncertainty about the product value (Goes, Karuga, and Tripathi, 2010 2012; Kagel and Levin, 1986). Consumers may then adapt their bidding strategies across sequential auctions and bid strategically in both auctions (Goes et al., 2012).

Simultaneous versus Sequential Auctions

Several papers have compared the bidding behavior and auction outcomes associated with~~of~~ both auction formats. In sequential auctions, bidders' learning results in higher revenue, but rational bidders, who anticipate this informational effect, also have an incentive to underbid. As a result, the seller's revenue may be higher or lower than in simultaneous auctions (Hausch, 1986). Betz et al. (2017) conducted an experiment comparing the revenues of simultaneous and sequential auctions. They found that sequential auctions resulted in higher revenues, which they attributed to fiercer competition for the item auctioned first.

[4] Cramton (1998) argues that simultaneous auctions provide price discovery, because bidders can process price information as the auction progresses. However, with snipe bidding, price information from ongoing auctions is not very useful in estimating final prices.

[5] Zeithammer (2006) conducted a series of analytical and empirical studies on the impact of forward-looking behavior on bidding behavior in sequential auctions. He empirically observed that bidders took information about future auctions into account, foresaw lower prices in future auctions, and reduced their bids when they knew about the future availability of products.

Commented [LG1]: Meaning: there would be ***fewer instances of aggressive bidding*** (less aggressive bidding) or ***bidding that is not as aggressive*** (less-aggressive bidding)? They are quite close in meaning, here, so you're fine either way.

~~However,~~Their results differed from those of~~different from~~ other ~~papers~~studies:~~,~~ they sold two different items (A and B, with different demand functions) and found that participants bought multiple units in both auctions, ~~such~~showing that they were not expected to be forward-looking bidders who shade their bids in the first auction. Haruvy and Popkowski Leszczyc (2018) studied simultaneous and sequential ascending auctions for identical products using different formats (regular ascending-bid auctions and voluntary-pay auctions—a specific type of all-pay auction in which all losing bidders are asked to voluntarily pay an amount equal to their high bid). They observed a significant amount of bidder switching between identical product auctions. (In addition, they found that bidders in voluntary-pay auctions more commonly used jump bidding and late entry.) Relative profitability (compared to the retail value) was higher in simultaneous auctions than in sequential auctions.

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Although previous research has considered bidders’ cross-bidding, learning, and/or forward-looking behavior in online simultaneous or sequential auctions, these two extreme auction formats neglect the general case of overlapping auctions: partial overlapping.

1.3. Partially Overlapping Auctions

The popularity of concurrent online auctions has spurred research on overlapping auctions. Bapna et al. (2009) analyzed auctions for identical electronic products from a major wholesaler’s online auction site, studying the impact of the degree of overlap, price information, and auction format on auction outcome. They found that auction overlap and price information about prior and following auctions all had a significant negative impact on ending prices, although the latter had a stronger influence. In particular, overlapping auctions attract more cross-bidders, who have a negative effect on auction prices.

Chang (2012, 2014) studied bidding strategies of heterogeneous bidders in overlapping English auctions from Sam’s Club. Both studies identified three different types of bidders (evaluators, opportunists, and participators), based on their bidding behaviors (i.e., the timing and the number of bids placed). Chang (2012) showed that as overlap (the total amount of time that two auctions are overlapped) increased, so did the proportion of opportunistic bidders (who monitor identical product auctions and cross-bid), resulting in lower auction prices. Chang (2014) focused on bidder learning and information revealed to bidders from other overlapping auctions. To capture information about different bidder strategies, he proposed a measure of entropy that is influenced by the degree of overlap and that affects auction revenue.

Haruvy et al. (2014) conducted a controlled field experiment, comparing simultaneous and partially overlapping auctions. ~~One condition involved~~They set up and analyzed, first, a pair of simultaneous auctions, then~~ other~~ a pair of partially overlapping (50% overlap in time) auctions for identical products. In addition, they manipulated the release of product information between products, to be either low or high. For "low information" only a picture of the product was provided, and for "high information" complete product information and a clear indicator of the value ~~are~~were provided. They did not find a difference in ending prices between full and partial overlapping auctions with the same level of information, though price dispersion was significantly higher in partial overlapping auctions due to less cross-bidding. More product information released had a negative effect on the ending price but did not affect the number of bidders or the number of cross-bids.

Elmaghraby et al. (2017) studied overlapping auctions for similar products: liquidating auctions for iPad tablets. In collaboration with a large wholesale liquidator for IT equipment, they conducted a field experiment in which they manipulated the starting prices of auctions. They found cross-product dependencies and that starting bids positively affected ending prices.

Han et al. (2018) studied overlapping eBay auctions for identical products and found that the reserve-price strategies of competing overlapping auctions influenced ending prices. This finding indicates that sellers should adapt their reserve-price strategies based on competitors' reserve-price strategies. (Competition is based on the number of overlapping auctions and the average level of competitors' reserve prices.) Results also have important implications for bidders regarding auction selection~~,~~ because auctions with low starting bids do not necessarily lead to lower selling prices, since they attract more bidders.

In summary, previous papers have indicated the importance of interdependency of concurrent auctions and addressed issues such as strategic bidding (bidder entry and cross-bidding), degree of overlap, bidder learning, and forward-looking behavior in concurrent auctions. However, previous research has mostly focused on sequential or simultaneous auctions and has not considered all the different issues in a single study. To address these concerns, we have developed a model of overlapping auctions, in a single framework, that includes bidder entry, cross-bidding, learning, forward-looking behavior, the degree of overlap and time discounting ~~in a single framework~~. Our results are consistent with some of the empirical finding discussed above. In particular, Bapna et al. (2009), and Chang (2012) find that an increase in auction overlap results in more cross-bidding and has a negative impact on ending prices. These results are consistent with our findings that a higher degree of overlap reduces the ability to learn and has a negative impact on ending prices. Also consistent with our discovery about the positive effect of learning, Pilehvar et al. (2017) found that bidder access to price information from concurrent and recently completed auctions for comparable products are significantly related to the ending price of the subsequent auctions.

1.4. Contributions

Our paper differs from the previous research as follows. Unlike most previous research, which has modeled auctions as stand-alone events, our study considers online concurrent auctions ***inter-dependently***. We propose that bids in concurrent auctions are linked via the actions of the bidders, their cross-bidding between auctions, their learning from previous auctions, and their forward-looking to future auctions. Bidders' cross-bidding~~g~~ between auctions increases the intensity of the competition, thus increasing the final bids in the auctions. Bidders' learning from past auctions reduces their valuation uncertainty, resulting in more aggressive bidding in future auctions. Finally, bidders being forward-looking,

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anticipating an opportunity to win the same product at a lower price in a future auction, results in bid shading in the current auction. Our analysis of bidders' behavior in concurrent auctions reveals complex interdependencies across concurrent auctions. We also show how the auction environment (e.g., degree of overlap, time discounting, and bidder entry) modifies bidder behaviors, weakening or strengthening the interdependencies across concurrent auctions.

Second, we model the degree of overlap ***endogenously*** as the seller's strategic response to bidders' forward-looking, learning, and cross-bidding behaviors. This approach ~~is~~ differ~~ent~~s from previous research that assumed the degree of overlap between auctions to be exogenous (i.e., they fix the degree of overlap to zero—sequential, half—partial overlapping, or full—simultaneous auctions), where bids are mostly decided by bidders' valuations, and the seller has very limited influence. In our study, the seller is able to decide the degree of overlap and thereby influence the accuracy of information provided to bidders.

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Third, we introduce uncertainty and explicitly incorporate ***bidder learning*** into the seller's decision-making (that is, at the end of the auction, the seller~~after~~ observ~~ing~~es the ending price and the bidding history)~~ into the seller's decision-making~~. Consumers, shopping online, often face uncertainties about the product value, because they are unable to touch/experience products before purchase. When they have a hard time assessing the quality of a product, observing the final bids in previous auctions may reduce their uncertainty. Including such bidder learning into our model makes results closer to reality.

Fourth, we identify and compare the forces (learning, forward-looking behavior, cross-bidding, bidder entry, and time discounting) that influence the seller's decision concerning overlapping strategies. Bidders in sequential auctions can be forward-looking and learn, bidders in simultaneous auctions can cross-bid, and bidders in partially overlapping auctions can do both. In our model, b~~B~~idder entry and learning have a negative effect on the overlap (resulting in the seller favoring sequential auctions), whereas forward-looking behavior and time discounting have a positive effect (resulting in the seller favoring simultaneous auctions).

With the coexistence of multiple forces, the optimal degree of overlap is derived by balancing these influences. For example, reducing auction overlap (increasing the combined duration of the auctions) results in increased *bidder entry*, boosting the intensity of competition among bidders and thereby the final bid, so running sequential auctions is optimal for the seller. Sequential auctions are also optimal for the seller when bidders can *learn* about product values from the preceding auction, because less overlap allows them to learn more about product values in the first auction. This learning results in more-aggressive bidding (and higher final bids) in both auctions: a direct effect in the second auction owing to reduced uncertainty through learning, and an indirect effect in the first auction, as bidders anticipate the impact of learning on the second auction. *Forward-looking* bidders shade their bid ~~shade~~ (bid less) in the current auction, anticipating a lower price in the future auction. Therefore, the seller wants total overlap to eliminate bid shading. Here, we extend previous work by showing how the degree of overlap affects the level of bid shading. *Time discounting* also has a positive effect on the overlap, because a seller wants to get paid earlier rather than later (preferring a shorter combined duration of the auctions). Knowing how those factors influence final bid prices helps the seller to better design their concurrent online auctions.

Last, we determine different regions for optimal auction formats. We find *sequential auctions* are optimal when valuation uncertainty ($\sigma_{\tilde{v}}^2$) is high and the impact of time discounting is weak (high β). ~~Because of~~When there is high- valuation uncertainty for bidders, sequential auctions give them ~~bidders have~~ ample opportunity to learn. Then, when the seller's profits are not heavily discounted, holding sequential auctions is optimal. ~~Under~~In this ~~condition~~situation, the influence from bidders' learning dominates. ~~On the opposite~~Conversely, *simultaneous auctions* are optimal when valuation uncertainty ($\sigma_{\tilde{v}}^2$) is low and the impact of time discounting is high (small β). ~~Under~~In this situation~~condition~~, the influence from bidders' forward-looking and time discounting behavior dominates; thus, the seller naturally wants to

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shorten the combined duration of the auctions and run them simultaneously. *Partially overlapping auctions* are optimal when valuation uncertainty is at a medium level, learning is not too difficult, and the ~~impact of~~ time discounting factor (the seller's desire to be paid quickly) is large. ~~Under~~In this ~~condition~~situation, the opposing effects acting upon the seller's revenue (the positive impact from time discounting and the negative impact from value uncertainty learning) are counterbalanced, such that neither dominates.

These results have important implications for retailers selling multiple identical items through auctions over time. By appropriately setting the degree of overlap for multiple auctions, auctioneers can increase their profitability. The managerial contributions are discussed in section 8.

The remainder of this paper is organized as follows. Section 2 sets up the framework of the analytical model. Section 3 provides analysis and results for the seller's decision without valuation uncertainty, and section 4 extends section 3 to the case of valuation uncertainty. Section 5 provides numerical analyses of our results. Section 6 presents a model extension with the rate of bidder entry, and section 7 a model where bidders are ~~allowed~~permitted to leave after the first auction. Finally, section 8 contains potential managerial contributions~~,~~ and limitations, and directions for future research.

2. Model Setup

Consider a firm selling two identical products in separate open-ascending online auctions. Each auction has an identical duration and a fixed ending time. The seller needs to determine the timing of the two auctions (i.e., should the items be sold simultaneously, sequentially, or partially overlapping?). The duration of an auction is set to one unit of time, and the degree of overlap is denoted as α (see Figure 1). When $\alpha = 0$, the seller holds *sequential* auctions; when $\alpha = 1$, he holds *simultaneous* auctions; when $1 > \alpha > 0$, he holds *partially overlapping* auctions.

Bidders want one product at most. We denote the number of bidders as n, where $n \geq 3$.[6] Bidders are rational and thus able to anticipate that the final price in the future auction will be lower (~~as~~ the bidder with the highest valuation will win in the first auction and will not participate in the second auction). As such, a bidder, with the opportunity to win an identical item in a future auction at a lower price, reduces her final bid in the current auction (Jofre-Bonet and Pesendorfer, 2003; Zeithammer, 2006, 2007). This reduction in the bid is called bid shading.

[6] For two overlapping auctions, at least three bidders need to be present. If only two bidders participate, each bidder will win one auction at a price equal to the minimum bid increment.

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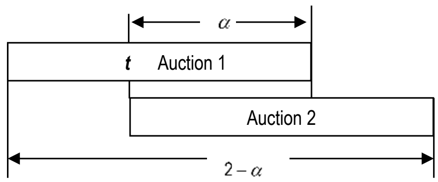

Figure 1. Overlapping auctions

We assume a model with bidders who face uncertainty concerning the value of the item (e.g., Lange et al., 2011; McGee, 2013).[7] This is common in online shopping where consumers only see a picture of the item and/or read a product description, which may be incomplete or in some cases even misleading (Zhuang, Popkowski Leszczyc, and Lin, 2018). This assumption is also consistent with a stream of research that has reported that consumers' WTP is influenced by what they observe and experience during the auction. Factors that influence their WTP include features of auction design such as starting bids and buy-it-now prices, and the number of bids and bidders (see the review by Haruvy and Popkowski Leszczyc, 2009). Therefore, we model bidder valuation $\tilde{v}_i$ as

$$\tilde{v}_i = v_i + \varepsilon_{\tilde{v}}, \tag{1}$$

where v_i denotes the true (unobserved) product value of bidder *i*, which differs across bidders. Valuations are distributed uniformly between [0, 1] with density function $f_v(.)$ and cumulative distribution function $F_v(.)$. $\varepsilon_{\tilde{v}}$ is the error term, which differs across bidders and is assumed to be drawn from a distribution with density function $f_{\tilde{v}}(.)$, cumulative distribution function $F_{\tilde{v}}(.)$, and mean 0 and variance $\sigma^2_{\tilde{v}}$. As a result, bidder *i*'s valuation $\tilde{v}_i$ is the combination of two independent distributions as illustrated in Figure 2.

[7] Differ~~ent~~ing from the standard private-values model in which consumers are assumed to know their own private values with certainty, these recent papers have relaxed this assumption and assume bidders have uncertain private values.

Figure 2 plots bidders’ distributions of valuations in three dimensions (X, Y, Z). The three axes meet at right angles to one another. The vertical Z-axis shows the probability of values on the X- and Y- axis. The horizontal X-axis shows expected valuation (mean). The Y-axis shows individual bidder~~’s~~s’ valuations. In the XY plane, the flat line $f_v(.)$ is the distribution of bidder’s expected valuations, which follows a uniform distribution U[0,1]. In the YZ plane, the small curves $f_{\tilde{v}_i}(.)$ are the distributions of bidders’ individual valuations, which vary around her expected valuation (mean). These distributions are assumed to be known to both bidders and the seller. The flat straight line of $f_v(.)$ is pe~~n~~rpendicular with the plane created by the curves of $f_{\tilde{v}_i}(.)$. Moreover, individuals’ valuations $f_{\tilde{v}_i}(.)$ are parallel to each other (except when bidders learn at the end of the first auction). That is, bidders do not learn from each other’s bidding during the bidding process. We assume that learning occurs after the final bids have been observed.

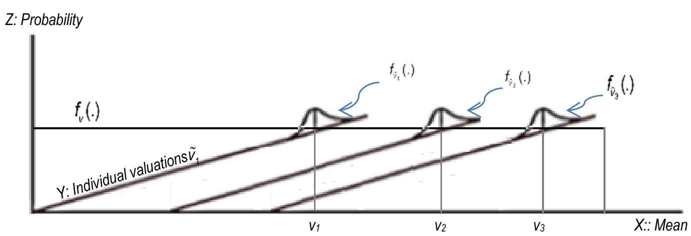

Figure 2. Distributions of bidders’ valuations

All bidders, except for the winner, bid up to their willingness to pay (WTP). We assume consumers are risk averse. When a bidder is certain about the product value, her WTP equals her valuation. When she feels uncertain about the product value, her WTP equals her valuation minus the risk premium. A higher

Formatted: Indent: First line: 0.63 cm level of uncertainty results in a lower WTP.[8] Therefore, bidder *i*'s WTP can be modeled as

$$WTP_i = \tilde{v}_i - r\sigma_{\tilde{v}}^2 / 2 \,, \tag{2}$$

where r denotes bidders' risk coefficient and $r\sigma_{\tilde{v}}^2 / 2$ is the risk premium.

Both the seller and the bidders discount their future payoff with discount rate β, which may vary between bidders, the seller, and across product categories (e.g., discount rates tend to be higher for highly depreciable tech or fashion products). Discounting measures the impatience or time value of the buyers and the seller. Seller and bidders can be either patient or impatient as the time discounting factor changes (see subsection 5.3). For a seller who sells items over time, the ability to sell more items simultaneously, versus fewer items sequentially, can provide a significant benefit, including increased revenues, lower cost of inventory, and lower depreciation of seasonal products. For consumers, there is considerable empirical evidence that bidders time discount prices or rewards. A number of experimental studies have found that bidders discount future rewards, even for auctions with a very short duration (Kirby,1997; Olivola and Wang, 2016). This is also consistent with several recent trends, like the popularity of "buy it now" option in online auctions, and same-day delivery by Amazon.com in certain cities. For simplicity, we assume discount rates to be the same for the seller and the bidders.

2.1. Learning

When bidders are uncertain about the product value, they tend to search for information to reduce their uncertainty. Part of this information is obtained from the bidding process, as bidders learn from the outcome of concurrent auctions. In the case of two overlapping auctions for identical products, we assume that bidders learn from the outcome of the first auction. In particular, they learn (receive a signal) from the final

[8]This mean-variance formulation has been widely used in finance studies (Pulley,1983) that have demonstrated it is a valid approximation of the Von Neumann-Morgenstern utility function. These existing studies illustrate that decision-makers can effectively maximize their expected utility when they only know the mean and the variance of their valuation distributions.

bids of individual bidders in the first auction. Therefore, the signal in this paper is the rank order of final bids of individual~~uvial~~ bidders in the first auction.

The game is played in four stages: (1) The seller determines the degree of overlap between two auctions. (2) Bidders join the first auction. When the second auction starts, bidders may continue to bid in the first auction or move to the second auction. (3) When the first auction ends, the winner gets the item, pays the amount of her bid, and leaves. The remaining bidders learn from the signal and update their valuations. (4) The remaining bidders bid in the second auction. At the end of the second auction, the winner gets the item and pays the amount of her bid. We are looking for the subgame perfect equilibrium.

We assume that at Stage 3 bidders learn from the final bids at the completion of the first auction, and not from the intermittent bids of individual bidders during the bidding process; as such, bidder's valuations are independent.[9] We use Equation 1 to model bidders' valuations at Stage 2 and Stage 4.

The major notations used in the paper are summarized in Table 2.

--- Insert Table 2 about here ---

3. Seller's Decision with Certain Product Valuation

We start with a model in which bidders are certain about the product value; that is, $\tilde{v}_i = v_i$. ~~w~~We solve the problem by backward induction. First, consider the bidders' behaviors at Stage 4. Because no further auctions follow, the second auction becomes a standard ascending-bid auction, and the bidder with the highest valuation among bidders wins the item and pays the price she bids. Therefore, the expected highest bid in the second auction is the expected second-highest valuation among the remaining $n-1$

[9] Bidders may also learn, during the auction, from the bids of other bidders. However, information obtained during the auction is limited. Initial bids are usually not very informative, because bidders tend not to bid up to their WTPs during their initial bids, and a significant proportion of bidders join and bid during the last few minutes of the auction (snipe bidding). Empirical evidence suggests over one third of bids arrive in the last few minutes of an auction, and as ~~such~~a result, many bidders do not reveal their WTP until the close of the auction (Roth and Ockenfels, 2002; Bajari and Hortascu, 2003). Therefore, learning at the end of the auction is far more informative than learning during the auction.

bidders (the winner of the first auction has left) (refer to Step 2 of Appendix A for the derivation); that is,

$$E[b_{(2)}^{[1]}] = E[v_{(2)}^{[2]}], \tag{3}$$

where the subscript in (.) denotes the running auction, superscript [ ] denotes the ranking of bidders' final bids/valuations in the running auction, and $E[.]$ denotes the expectation. Hence, $b_{(2)}^{[1]}$ refers to the highest bid in the second auction and $v_{(2)}^{[2]}$ refers the second-highest bidder valuation in the second auction.

Next, we consider Stage 2 (we omit Stage 3 because no learning occurs when bidders are certain about the product value). In this stage, all bidders bid in the first auction. The expected utility for the rational forward-looking bidder *i*, if she wins in the second auction, is

$$E[u_i] = E[v_i] - E[b_{(2)}^{[1]}].$$

The expected third-highest bidder valuation in the first auction equals the expected second-highest bidder valuation in the second auction, $E[v_{(1)}^{[3]}] = E[v_{(2)}^{[2]}]$, because the winner of the first auction leaves and the remaining bidders participate in the second auction. Combining Equation 3 with $E[v_{(1)}^{[3]}] = E[v_{(2)}^{[2]}]$, we have her utility as

$$E[u_i] = E[v_i] - E[v_{(1)}^{[3]}]. \tag{4}$$

Then, when she bids in the first auction, considering her utility if winning in the second auction, she shades (reduces) her final bid in the first auction to a level that makes her indifferent ~~between~~to both possibilities: winning in the first or in the second auction. That is,

$$E[v_i] - E[b_{i(1)}] = \beta^{1-\alpha} E[u_i], \tag{5}$$

where LHS is her expected utility when winning in the first auction and RHS is her discounted utility when winning in the second auction. Substituting Equation 4 into 5, we have her expected bid in the first auction:

$$E[b_{i(1)}] = (1-\beta^{1-\alpha})E[v_i] + \beta^{1-\alpha} E[v_{(1)}^{[3]}]. \tag{6}$$

Next, we show the rationale that the final bid in the FIRST auction equals the final bid (adjusted for bid shading) of the bidder with the second-highest valuation. In the FIRST auction bidders with low valuations gradually drop out as the bid level surpasses their valuation, and, therefore, they do not shade their final bid. When the bid level is equal to the third-highest valuation, the bidder with the third-highest valuation drops out and only two bidders remain. These bidders know that they are able to win the product, either in the first auction or in the second. Therefore, they shade their final bid in the first auction, and bid up to the point which makes them indifferent ~~between~~to whether they win~~ning~~ in the FIRST ~~and~~or the SECOND auction.

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From Equation 6, we know the indifference point for the highest valuation bidder,

$$E[b_{1(1)}] = (1-\beta^{1-\alpha})E[v_{(1)}^{[1]}] + \beta^{1-\alpha}E[v_{(1)}^{[3]}].$$

If the current high bid is higher than $E[b_{1(1)}]$, the bidder will drop out of the FIRST auction and moves to the SECOND auction. We also know that the point of indifference for the second-highest valution bidder is

$$E[b_{2(1)}] = (1-\beta^{1-\alpha})E[v_{(1)}^{[2]}] + \beta^{1-\alpha}E[v_{(1)}^{[3]}]$$

When the bid level reaches $E[b_{2(1)}]$, the bidder with the second-highest valuation drops out of the FIRST au~~c~~c~~u~~tion, since $E[b_{2(1)}] < E[b_{1(1)}]$. In this case, the highest valuation bidder wins in the FIRST auction, and her expected payment is $E[b_{(1)}^{[2]}]$. That is,

$$E[b_{(1)}^{[1]}] = (1-\beta^{1-\alpha})E[v_{(1)}^{[2]}] + \beta^{1-\alpha}E[v_{(1)}^{[3]}] \tag{7}$$

Now we analyze bid ~~-~~shading Δ_α. Without forward-looking behavior, the winner bids up to $E[v_{(1)}^{[2]}]$ in the first auction (see Step 2 of Appendix A for details); with forward-looking behavior, the winner bids up to $b_{(1)}^{[1]}$ in the first auction (see Equation 7). Clearly, a forward-looking winner pays less in the first auction, and the difference is the amount of bid shading and is expected to be:

$$\Delta_{\alpha} = \beta^{1-\alpha}(E[v_{(1)}^{[2]}] - E[v_{(1)}^{[3]}]).$$

Given that bidder valuations follow a uniform distribution from [0, 1], the expected amount of bid shading is rewritten as

$$\Delta_{\alpha} = \beta^{1-\alpha} / (n+1).$$

Finally, we consider stage 1, in which the seller decides the degree of overlap. The seller's revenue R is the sum of the final bids of both auctions, derived in Equations 3 and 7, respectively. Therefore, $E[R] = E[b_{(1)}^{[1]}] + \beta^{1-\alpha} E[b_{(2)}^{[1]}]$, which is simplified to

$$E[R] = (n-1)/(n+1) + \beta^{1-\alpha}(n-3)/(n+1).$$

We also need to look at the extreme case in which two auctions run simultaneously ($\alpha = 1$). When auctions run simultaneously, bidders are not forward-looking; however, they can cross-bid, bidding in the auction with the lowest bid level. As a result, one auction ends with the final bid at the second-highest valuation, and the other auction ends with the final bid at the third-highest valuation. The seller's expected revenue is $E[v^{[2]}] + E[v^{[3]}]$, which is simplified as $(2n-3)/(n+1)$ (refer to Appendix A for details).

The expected bid-shading and the seller's revenue are summarized as follows:

$$\Delta_{\alpha} = \begin{cases} \beta^{1-\alpha} / (n+1) & 0 \le \alpha < 1 \\ 0 & \alpha = 1 \end{cases}, \tag{8}$$

$$E[R] = \begin{cases} (n-1)/(n+1) + \beta^{1-\alpha}(n-3)/(n+1) & 0 \le \alpha < 1 \\ (2n-3)/(n+1) & \alpha = 1 \end{cases}. \tag{9}$$

Equations 8 and 9 show what happens when two independent auctions become linked through bidders' forward-looking behavior. Forward-looking behavior results in a lower final bid in the first auction, thus reducing the seller's profits. Equation 8 depicts the trajectory of the amount of bid-shading as the degree of overlap changes.

Note that $\partial \Delta_{\alpha} / \partial \alpha > 0$; that is, the amount of bid-shading becomes larger as auction overlap increases until α reaches 1. The reason is as follows: When the overlap increases, the total auction duration is reduce~~s~~d, which increases the future payoff to the bidder (with the second-highest valuation). As ~~such~~a result, she would shade her final bid more in the first auction (see Figure 3-1). Also note that $\partial\Delta_{\alpha} / \partial n < 0$; that is, the amount of bid-shading decreases as the number of bidders increases. The increase in the number of bidders intensifies the competition among bidders, reducing the chance to win in either auction. As such, the bidder (with the second-highest valuation) shades her bid less in the first auction (see Figure 3-2).

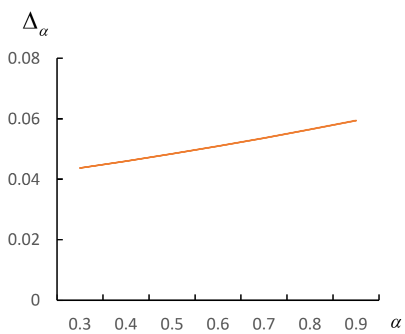

Figure 3-1 The expected bid-shading under different degrees of overlap (for β = 0.6)

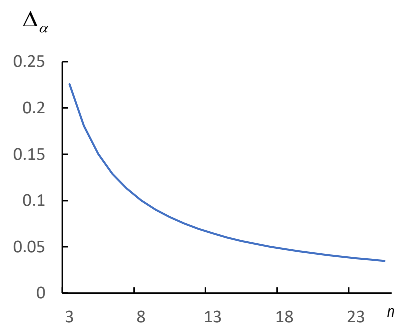

Figure 3-2 The expected bid-shading for different number of bidders (for β = 0.6)

Comparing the two equations for the seller's expected revenue in Equation 9 shows the expected revenue with full overlap is larger than that with any other degree of overlap (see Figure 4). Hence, it is optimal for a seller to run simultaneous auctions, because this strategy can eliminate the loss caused by bidders' forward-looking behavior and discounting, which is consistent with empirical findings derived from eBay data (see Zeithammer, 2006). We summarize this finding in Proposition 1.

PROPOSITION 1. *Running auctions simultaneously ($\alpha = 1$) is an optimal strategy for a seller when bidders know product values and product values are distributed uniformly.*

By increasing the degree of overlap, the seller reduces the extent of bid shading and time discounting, which both have a negative effect on revenue. In addition, because the number of bidders is fixed and bidders know the value of the product, the seller cannot increase the number of bidders or facilitate learning by reducing auction overlap. Therefore, the seller benefits from increasing the overlap and running auctions simultaneously.

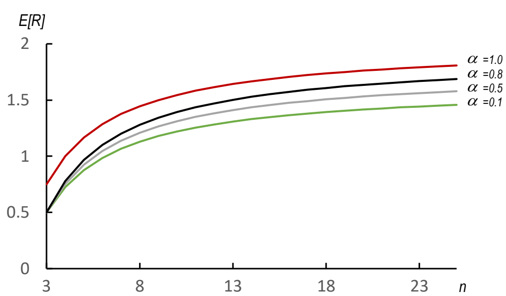

Figure 4. Seller's expected revenue under different degrees of overlap (for β = 0.6)

4. Seller's Decision with Uncertain Product Valuation

A common issue for online purchases is that bidders are uncertain about product values~~,~~ because consumers cannot inspect and/or experience products. We next consider the seller's optimal overlapping strategy in this situation~~under this condition~~, whe~~re~~n bidders have uncertain valuations ~~and~~but they can learn about the product value from the bidding histories of concurrent auctions, which end before the focal one.

4.1. Bayesian Learning

We propose a model in which bidders revise their beliefs using Bayesian updating. When bidders are not certain about the value of an item, they can learn from the outcomes of previous auctions (i.e., how other bidders value the product). ~~-~~As mentioned above, bidders learn from the outcome of the first auction. In particular, they observe a signal *s*, which consists of the rank order of the final bids of all bidders.

We assume bidders are Bayesian learners who have an expectation of the product value, $E[\tilde{v}_i]$ (the prior information), and form their posterior expectation after receiving signal s (the new information). We denote the density of signal s as $f_{s|\tilde{v}_i}$, which is conditional on the distribution of $\tilde{v}_i$, and bidder i's posterior density of $\tilde{v}_i \mid s$, denoted as $f_{\tilde{v}_i|s}$.

Applying Bayes' rule, bidder i updates her valuation upon receiving signal s by

$$f_{\tilde{v}_i|s}(\tilde{v}_i \mid s) = \frac{f_{s|\tilde{v}_i}(s \mid \tilde{v}_i) f_{\tilde{v}_i}(\tilde{v}_i)}{f_s(s)} , \tag{10}$$

with $f_s(s) = \int f_{s|\tilde{v}_i}(s \mid \tilde{v}_i) f_{\tilde{v}_i}(\tilde{v}_i) d\tilde{v}_i$.

When the prior and new information both follow Gaussian distributions, Bayes' rule shows the posterior is a function of the variances of the prior and signal s, as follows:

$$E[\tilde{v}_i \mid s] = \frac{1/\sigma_{\tilde{v}_i}^2 E(\tilde{v}_i) + 1/\sigma_s^2 E(s)}{1/\sigma_{\tilde{v}_i}^2 + 1/\sigma_s^2} . \tag{11}$$

Let us define τ , the precision of signal s, as

$$\tau = \frac{1/\sigma_s^2}{1/\sigma_{\tilde{v}_i}^2 + 1/\sigma_s^2} . \tag{12}$$

Inputting Equation 12 into Equation 11, we then have the learning equation:

$$E[\tilde{v}_i \mid s] = \tau E[s] + (1-\tau) E[\tilde{v}_i] . \tag{13}$$

Note the weights on the signal and the prior are not fixed and depend on the level of confidence ~~on~~in the prior and the signal. When bidders have high confidence in their prior, they put less weight on the signal, and any changes in valuations are small. When the signal is more informative, bidders put more weight on it, and any changes in valuations are large. Hence, adjustments in valuations depend on the level of the variance of the prior relative to that of the signal.

We next investigate the effect of the signal on the mean and the variance of the posterior. Taking the expectation on the posterior mean, we have:

$$E[E[\tilde{v}_{\boldsymbol{i}} \mid s]] = \int (\int \tilde{v}_{\boldsymbol{i}} f_{\tilde{v}_i|s}(\tilde{v}_{\boldsymbol{i}} \mid s) d\tilde{v}_{\boldsymbol{i}} f_s(s) ds \,. \tag{14}$$

Inputting Bayesian rule (10) into (14), we have

$$E[E[\tilde{v}_{\boldsymbol{i}} \mid s]] = \int \int f_{s|\tilde{v}_i}(s \mid \tilde{v}_{\boldsymbol{i}}) \tilde{v}_{\boldsymbol{i}} f_{\tilde{v}_i}(\tilde{v}_i) ds d\tilde{v}_{\boldsymbol{i}} \,. \tag{15}$$

As $\int f_{s|\tilde{v}_i}(s \mid \tilde{v}_i) ds = 1$ for any $\tilde{v}_i$, we then simplify Equation 15 as

$$E[E[\tilde{v}_{\boldsymbol{i}} \mid s]] = \int \tilde{v}_{\boldsymbol{i}} f_{\tilde{v}_i}(\tilde{v}_i) d\tilde{v}_{\boldsymbol{i}} = E[\tilde{v}_{\boldsymbol{i}}] \,. \tag{16}$$

The ~~above~~ equation above shows that Bayesian learning satisfies the martingale property, where the mean of the posterior is expected to be the same as the mean of the prior. This central property of Bayesian learning has been ~~proofed~~proved in statistics and economics (e.g., Gentzkow and Kamenica, 2011 (p2594); Chamely, 2003 (p35)). This implies the expected value of the revision is zero.

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We next consider the variance of the posterior. According to Bayesian rule, it is

$$\sigma^2_{\tilde{v}_i|s} = \frac{\sigma_s^2 \sigma_{\tilde{v}_i}^2}{\sigma_s^2 + \sigma_{\tilde{v}_i}^2} \,. \tag{17}$$

Inputting Equation 12 into 17, we then have

$$\sigma^2_{\tilde{v}_i|s} = (1-\tau)\sigma^2_{\tilde{v}_i} \,. \tag{18}$$

The above equation shows the martingale convergence theorem, which indicates that when receiving more information, Bayesian beliefs converge.

Therefore, we showed that the mean of the posterior is expected to be the same as the mean of the prior (see Equation 16), and that the variance of the posterior is smaller than the prior (see Equation 18), resulting in the following proposition.

PROPOSITION 2. *The mean of the posterior is expected to be the same as that of the prior, and the variance of the posterior is smaller than that of the prior.*

Proposition 2 reveals that after learning occurs, the expected mean of the product value does not

Formatted: Indent: First line: 0.63 cm change, but bidders' uncertainty reduces. As ~~such~~a result, bidders may bid more aggressively in the second auction. Moreover, if the signal is more informative (i.e., a larger τ) and the difference between the bidder's prior information and the signal is larger, the update of the posterior will be larger. In Figure 2, the small curves mounted on the flat line become narrower after learning.

4.2. Impact of Overlap on Learning

Bidders' learning is influenced by the precision of the signal and varies by product category. For some products, such as experience products, artwork, and new products, assessing the value is difficult. Therefore, we denote k ($\in (0,1)$) as the ease of learning for different product categories, where a small k accounts for product categories with a high difficulty in learning the product value.

The degree of overlap influences learning through the precision of the signal. In *sequential (non-overlapping) auctions*, the final bid of individual bidders reflects their valuation (WTP), as all bidders in English auctions (except the winner) bid up to their WTP (Cramton 1998). Therefore, the signal, consisting of the final bid of individual bidders, reflects bidders' valuation well. In *simultaneous auctions,* bidders cannot learn from the completed bidding history of the other auction, hence there is no signal.

In *partial overlapping auctions*, bidders in the second auction observe the completed bidding history from the first auction, and they have enough time to adjust their bids in the second auction accordingly. However, the degree of overlap influences the final bid (WTP) in both auctions because of cross-bidding and bid- shading.

Cross-bidding. In partially overlapping auctions, bidders can move from the first to the second auction, before bidding up to their WTP in the first auction. (Note that since prices are lower in the second auction, some bidders move to the second auction but do not switch back to the first auction.) Suppose that the second auction starts at time *t, as* shown in Figure 1. For ~~B~~bidders who drop out of the first auction *before t,* their final bid reflects their WTP. Bidders who drop~~s~~ out *after t* can move from the first to the second auction

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before bidding up to their WTP in the first auction. ~~As such,~~Having this option means that their final bid in the first auction is a poor reflection of their WTP. The larger the overlap, the greater the proportion of bidders who move and thus the less precise the signal becomes. Hence, the quality of the signal is negatively related to the degree of overlap.

Bid~~-~~ shading. With the availability of the second auction, the bidder with the second-highest valuation shades her final bid in the first auction. As shown in Section 3, Figure 3-1, the expected amount of bid-shading is positively related to the degree of overlap, because of the time-discounting effect. That means that her final bid in the first auction is negatively influenced by the overlap. Hence, the quality of the signal from which bidders learn is negatively related to the degree of overlap.

In summary, the precision of the signal is decreasing in the degree overlap, most informative at zero overlap and least at full overlap. We therefore model the precision of the signal as a function of both the product category and the degree of overlap as follows[10]:

$$\tau = k(1-\alpha)^{1/2}. \tag{19}$$

From this equation, we can see $d\tau / d\alpha < 0$. At the one extreme, when auctions are held sequentially without any overlap ($\alpha = 0$), $\tau = k$. In this case, the precision of the signal is influenced by the product category, but not by cross-bidding. At the other extreme, when auctions are held simultaneously with full overlap ($\alpha = 1$), $\tau = 0$, the precision of the signal is lowest, as cross-bidding is at its maximum and bidders learn the least. Therefore, by varying the degree of overlap, the seller can influence the precision of the signal and thus the ability ~~for~~of bidders' to learn~~ing~~ from the first auction.

4.3. Seller's Overlapping Decision

[10] Equation 19 is a simplified model that captures the negative relationship between the degree of overlap and the precision of the information. In Web Appendix C, we provide a discussion of the relationship between the degree of overlap and the precision of the signal. An additional simulation is provided for a linear relationship for the precision of the signal.

We discuss the cases of $\alpha = 1$ and $\alpha \neq 1$ separately. For the case of $\alpha = 1$ (simultaneous auctions), bidders cannot be forward-looking, and thus no bid shading happens. Moreover, because the two auctions end simultaneously, no learning happens. As bidders can switch across auctions, the second-highest valuation bidder, if she bids out in one auction, can switch to the other auction to win. Therefore, one auction ends with the final bid at the second-highest valuation, and the other auction ends with the final bid at the third-highest valuation. The seller's expected revenue, based on Equation 2, is $E[\tilde{v}^{[2]}] + E[\tilde{v}^{[3]}] - r\sigma_{\tilde{v}}^2$, which further equals $E[v^{[2]}] + E[v^{[3]}] - r\sigma_{\tilde{v}}^2$, where the first two terms are derived as follows: Taking the expectation on both sides of Equation 1 yields the individual bidder's expected valuation equal to her product value, because the mean of the bidder's estimation error is zero. Moreover, we derive $E[v^{[2]}] = (n-1)/(n+1)$ and $E[v^{[3]}] = (n-2)/(n+1)$ (see Appendix A for the technical details). As product value is assumed to follow a uniform distribution in the support of [0, 1], the seller's expected revenue can be further simplified to $(2n-3)/(n+1) - r\sigma_{\tilde{v}}^2$.

For the case of $\alpha \neq 1$ (partially overlapping auctions), bidders can be forward-looking and learn. We solve this four-stage game (specified in the sequence of the game of section 2) via backward induction. The seller's overlapping decision is summarized as follows (refer to Appendix B for the proof).

PROPOSITION 3. *When bidders are uncertain about their valuations, and product values are distributed uniformly,*

1. *The amount of bid shading is* $\Delta_\alpha = \begin{cases} \beta^{1-\alpha}/(n+1) + [(1-\tau)\beta^{1-\alpha} - 1]\sigma_{\tilde{v}}^2/2 & 0 \le \alpha < 1 \\ 0 & \alpha = 1 \end{cases}$ *and*

   the seller's expected revenue is $E[R] = \begin{cases} \left(n-1+\beta^{1-\alpha}(n-3)\right)/(n+1) - \beta^{1-\alpha}(1-\tau)r\sigma_{\tilde{v}}^2 & 0 \le \alpha < 1 \\ (2n-3)/(n+1) - r\sigma_{\tilde{v}}^2 & \alpha = 1 \end{cases}$.

2. *A unique degree of overlap* $\alpha^* \in [0,1]$ *exists that maximizes the seller's expected revenue in two auctions selling identical products. In particular, the optimal degree of overlap is as follows:*

   (1) When time discounting exists (i.e., $\beta \neq 1$*),*

   if $\left((1-k) - (n-3)/\left((n+1)r\sigma_{\tilde{v}}^2\right)\right)\ln\beta < k$ *, then* $\alpha^* = \tilde{\alpha}$;

if $\left((1-k)-(n-3)/\left((n+1)r\sigma_{\tilde{v}}^2\right)\right)\ln\beta \geq k$*, then if* $\beta > \tilde{\beta}$, $\alpha^* = 0$; else $\alpha^* = 1$, where $\tilde{\alpha}$ satisfies $\left(\left(1-k+k\alpha\right)-(n-3)/\left((n+1)r\sigma_{\tilde{v}}^2\right)\right)\ln\beta = k$ and $\tilde{\beta} = \left((n-2)-(n+1)\left(1-\beta(1-k)\right)r\sigma_{\tilde{v}}^2\right)/(n-3)$.

(2) When time discounting does not exist (i.e., $\beta = 1$*),*

if the risk premium is sufficiently large (i.e. $r\sigma_{\tilde{v}}^2 > 1/(n+1)$*), running sequential auctions (*$\alpha^* = 0$*) is optimal; otherwise, running simultaneous auctions (*$\alpha^* = 1$*) is optimal.*

The first part of Proposition 3 provides the extent of bid shading by bidders and the related revenue for the seller. The second part of Proposition 3 shows the optimal degree of overlap under different market conditions with and without time discounting.

When time discounting does exist, we will show in the next section the conditions ~~when~~under which overlapping auctions are optimal, which is a function of valuation uncertainty ($\sigma_{\tilde{v}}^2$), time discounting (β), ease of learning (k), the intensity of the competition (n), and risk attitudes (r). (In section 5, we conduct simulation analyses to further investigate the impact of these variables on overlapping strategies.)

When time discounting does not exist as an option, it is optimal for the seller to run auctions sequentially when bidders' risk premium is sufficiently high (i.e., $r\sigma_{\tilde{v}}^2/2 > 1/2(n+1)$) and run auctions simultaneously otherwise (see Lemma 1 in Appendix B for the proof). This result is driven by two opposing forces on the degree of overlap: a negative effect due to bidder learning (the seller gains owing to more-aggressive bidding in the future auction as bidders learn, and thus he prefers to reduce the degree of overlap) and a positive effect from forward-looking behavior (the seller loses owing to bidders' bid shading in the first auction, and thus he prefers fully overlapping auctions). Depending on which force dominates, the seller prefers to run auctions either sequentially or simultaneously.

We also consider the case in which time discounting ~~exists~~is an option for the seller ($\beta_{seller} \neq 1$) but not

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or
When time discounting is not possible, … (or is not selected by the seller, …; or is not desired, …)

for consumers ($\beta_{bidder} = 1$).[11] We observe that the seller prefers sequential auctions when uncertainty is high and the discounting effect is weak. ~~Moreover~~However, seller profits decline, because bidders increase bid shading in the first auction (see Web Appendix B for the proof).

Finally, we consider different degrees of risk sensitivity of bidders. Like most auction research, our results in Proposition 3 are based on the frequently used assumption that bidders are risk averse (Haruvy and Popkowski Leszczyc, 2009). Risk aversion has also been attributed to bidders who overbid~~ding~~ in some auctions to reduce the likelihood of losing the item they bid on at auction. However, bidders may be risk ~~-~~neutral or risk-takers.

We start with the case in which bidders are risk neutral; that is, $r = 0$. Then, our model with learning becomes the same as the model without learning, because the bidding function $WTP_i = \tilde{v}_i - r\sigma_{\tilde{v}}^2 / 2 = \tilde{v}_i$. As a result, the optimal degree of overlap is full overlap.

Second, when bidders are risk-takers, $r < 0$, they like uncertainty, and bid high when the uncertainty is high, and bid low when the uncertainty is low. ~~Then~~This means that the seller does not benefit from learning by bidders who are risk-takers~~,~~ because learning reduces their uncertainty. Hence, learning reduces the final bid in the second auction. Since~~,~~ learning, time discounting, and forward-looking behavior all reduce the final bids in the auctions, the seller prefers a greater degree of overlap, and thus full overlap is optimal (see Appendix C for the proof).

5. Numerical Analyses: Results and Insights

The purpose of the numerical analyses is to illustrate the findings in Proposition 3 and to examine how parameters $\{n, r, \sigma_{\tilde{v}}^2, \beta, k\}$ in our model influence the seller's profitability and optimal overlapping strategy

[11] Sellers who need to sell multiple items over time, and who carry a significant inventory level, tend to be more sensitive to discount rates (especially for time-sensitive goods, like seasonal items or high-tech products).

(because we got clear outcomes for the condition without time discounting, we focus here only on conditions with time discounting). Valuation uncertainty ($\sigma_{\tilde{v}}^2$) varies from 0.1 to 0.9,[12] and time discounting (β) varies from 0.5 to 0.9. Ease of learning (k) varies from 0.1 to 0.9, since the precision of the signal τ is assumed to be in (0, 1), k also has to be in (0, 1), because $\tau = k(1-\alpha)^{1/2}$, and $\tau = k$ when $\alpha = 0$. The intensity of the competition (or the number of bidders n) has three different levels (5, 8, and 11 bidders).[13] Finally, risk attitudes (r) are set to 0.8, ~~such~~indicating that bidders are risk averse.

Tables 3a and 3b show the optimal degree of overlap and seller profits for different values of ease of learning (k), valuation uncertainty ($\sigma_{\tilde{v}}^2$), and time discounting factor (β) for the number of bidders n = 11 and risk attitude $r = 0.8$. The tables report only part of the parameter values, which is sufficient to assess the impact of the different variables on the seller's profitability and overlapping strategies (e.g., the regions where different overlapping strategies dominate). The full results are available upon request.

--- Insert Tables 3a and 3b about here ---

Overall results indicate simultaneous auctions are optimal when the ease of learning (k) and valuation uncertainty ($\sigma_{\tilde{v}}^2$) are low and time discounting is high (smaller β). Under these conditions, where the effect of time discounting and forward-looking behavior dominate the effect of learning, the seller wants to reduce the overall duration of the auctions and run them simultaneously.

When bidders have ample opportunity to learn because of high-valuation uncertainty and when the effect of time discounting is low (larger β), holding sequential auctions is optimal for the seller. Under these

[12] This is the calculation for the variation ~~B~~because product valuations follow a uniform distribution in [0, 1].

[13] We only report the results based on a total of 11 bidders in both auctions. We also conducted a numerical analysis for n = 8 and n = 5. In general, we find that as the number of bidders increases, the optimal degree of overlap increases. The difference between n = 8 and n = 11 is very small, whereas for n = 5, we see a larger number of occurrences in which sequential auctions are optimal.

conditions, the seller wants to increase the combined duration of the auctions such that bidders have a greater opportunity to learn.

Partially overlapping auctions are optimal when valuation uncertainty is at a medium level, the learning is not too difficult, the effect of time discounting is medium~~,~~; this is the region where the positive impact from time discounting and forward-looking strategies matches the negative impact from learning.

5.1. The Influence of Ease of Learning

k is the parameter accounting for the ease with which bidders learn from the signal *s*. A larger value of *k* implies bidders can learn more easily, which reduces valuation uncertainty after receiving the signal. The first section of Table 3a shows the impact of *k* for different levels of $\sigma_{\tilde{v}}^2$ and β (to illustrate these trends Web Appendix D separates the first section of Table 3a into Tables WD1 and WD2). Overall, the ease of learning *k* has little impact on the optimal degree of overlap when valuation uncertainty is either high or low. When valuation uncertainty is low ($\sigma_{\tilde{v}}^2$ = 0.1), bidders have no room to learn, and running auctions *simultaneously* is most~~ly~~ optimal (to reduce the negative impact of time discounting and forward-looking behaviour). When valuation uncertainty is high ($\sigma_{\tilde{v}}^2$ = 0.9) and bidders benefit from learning, running auctions *sequentially* is mostly optimal**,** allowing bidders to learn the most. The ease of learning *k* has an influence only when time discounting is at a medium level, ~~where~~in which case *partially overlapping* auctions become optimal.

The ease of learning *k* has no impact on the seller's profit when valuation uncertainty is low. When valuation uncertainty is low ($\sigma_{\tilde{v}}^2$ = 0.1) and the time discount rate is high (low β), the sellers should run auctions *simultaneously* to reduce the negative impact of time discounting and forward-looking behavior. When valuation uncertainty is high ($\sigma_{\tilde{v}}^2$ = 0.9) and the time discount rate is low (high β), the WTP in auctions is lower. Although the seller runs sequential auctions to reduce the uncertainty, profits still suffer

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from time discounting of the final bid in the second auction and from bid ~~-~~shading in the first auction. In general~~ly~~, the

seller's profit increases with the ease of learning.

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5.2. The Influence of Valuation Uncertainty

Bidders~~,~~ who face uncertainty about the product value~~,~~ can learn about the value after the first auction ends. Learning reduces uncertainty; thus, bidders bid more aggressively in the second auction, and the final bid increases. Meanwhile, anticipating the bid increase in the second auction, they may also increase their bids in the first auction.

The second section of Table 3a shows the impact of $\sigma_{\tilde{v}}^{2}$ for different levels of *k* and β (t~~T~~o further illustrate these trends Web Appendix D separates the second section of Table 3a into Tables WD3 and WD4). We observe that valuation uncertainty has a negative impact on the ~~optimal~~ degree of overlap. For low values of uncertainty ($\sigma_{\tilde{v}}^{2} = 0.1$), running auctions *simultaneously* is optimal when the ease of learning is low (it is difficult to learn) and the time discount rate is high (low β), because the opportunity to learn is the lowest and time discounting and bid ~~-~~shading dominate all factors. For high values of valuation uncertainty ($\sigma_{\tilde{v}}^{2}$ = 0.9), running auctions *sequentially* is optimal when the ease of learning is high and the time discount rate ~~-~~is low (high β), ~~such~~so that bidders can learn the most (which effect dominates all factors). Finally, for the medium levels of valuation uncertainty, as long as the time discount rate is not too low, *partially overlapping* auctions are most~~ly~~ optimal.

Commented [LG14]: Optimal degree of overlap or just degree of overlap?

Valuation uncertainty has a negative impact on the seller's profit, as bidders bid less as uncertainty increases. Profit is the lowest in the region where valuation uncertainty is the highest ($\sigma_{\tilde{v}}^{2}$ = 0.9),~~-~~ the ease of learning is the smallest (k=0.1) and the discount is the strongest (β =0.5). Reduced profit is caused by the combination of time discounting, valuation uncertainty, bid shading and lack of learning. ~~-~~In the region where valuation uncertainty is the lowest ($\sigma_{\tilde{v}}^2$ = 0.1), the ease of learning the highest (k=0.9) and the time discount the lowest (β =0.9), profit is the highest. In general, the seller's profit decreases with the increase in valuation uncertainty.

5.3. The Influence of Time Discounting

The effect of time discounting is incorporated through $\beta^{1-\alpha}$, which is inversely related to the discount factor β. Hence, a lower value for β implies a higher discount rate. Table 3b displays the impact of β for different levels of *k* and $\sigma_{\tilde{v}}^2$ (to further illustrate these trends Web Appendix D separates Table 3b into Tables WD5 and WD6).

Results show that, on average, greater time discounting (β is low) results in a higher degree of overlap. Because time discounting reduces the seller's revenue, increasing the degree of auction overlap and thus shortening the combined duration of the auctions is optimal. However, when uncertainty is low and consumers are offered little opportunity to learn, running *simultaneous* auctions is optimal; when uncertainty is high and consumers can learn a lot, running *sequential* auctions is optimal.

Time discounting has a negative impact on the seller's profit. The reduced profit is caused by the combination of time discounting, valuation uncertainty, bid shading and lack of learning. Profit is lowest when valuation uncertainty is the highest ($\sigma_{\tilde{v}}^2$ = 0.9), the ease of learning is the smallest (k=0.1) and the discount is the strongest (β =0.5) (see left-upper corner of the right subsection in Table 3b). Profit is highest when valuation uncertainty is the lowest ($\sigma_{\tilde{v}}^2$ = 0.1), the ease of learning the highest (k=0.9) and the time discount the lowest (β =0.9) (see the right-upper corner of the left subsection in Table 3b). In general, the seller's profit decreases with the increase in time discounting effect.

6. Extension: Bidder Entry

In online auctions, bidders are often allowed to join an auction during the bidding session. Hence, we relax the assumption that the number of bidders is fixed (for both auctions). We use a Poisson process to model the arrival of bidders, and we denote the average arrival rate as η. Bidders who arrive before the end of the first auction are called “initial” bidders, and those who arrive afterwards are called “new” bidders. These two segments of bidders are assumed to differ in their valuation distributions when they compete in the second auction: Initial bidders have observed the bidding result of the first auction and thus updated their valuations, but new bidders have not. New bidders who arrive in the second auction most likely do not learn from the outcome of the first auction. Bidders who are new to the website are not aware of the first auction. While bidders may be able to search for completed auctions, less experienced bidders are less likely to use all information available on the website (Wilcox, 2000).

As previously presented, we let the superscript in [ ] denote the ranking of valuations. Then $Initial^{[i]}$ refers to the i^{th}-highest WTP among *initial* bidders, and $New^{[i]}$ refers to the i^{th}-highest WTP among *new* bidders. By the end of the second auction, depending on the possible rank order in valuation among bidders in the second auction, one of three cases occurs:

(1) An *initial* bidder wins and pays the price of $Initial^{[3]}$ if $Initial^{[3]} > New^{[1]}$, where the top two bidders' valuations in the second auction are $Initial^{[2]}$ and $Initial^{[3]}$;

(2) An *initial* bidder wins and pays the price of $New^{[1]}$ if $Initial^{[2]} > New^{[1]} > Initial^{[3]}$, where the top two bidders' valuations in the second auction are $Initial^{[2]}$ and $New^{[1]}$;

(3) A *new* bidder wins and pays the price of $Initial^{[2]}$ if $New^{[1]} > Initial^{[2]}$, where the top two bidders' valuations in the second auction are $New^{[1]}$ and $Initial^{[2]}$.[14]

High ~~B~~bidder entry tends to benefit the seller for all the cases above ~~cases~~. In case 1, the seller is better off because there is no bid shading in this extension, as the *initial* bidder with the second~~-~~highest valuation will not bid shade in the first auction, because she fears that if she loses the first auction, she has

[14] The case $New^{[2]} > Initial^{[2]}$ never happens, because simple calculation shows the expected second-highest WTP among the *initial* bidders always exceeds the expected second- highest WTP among the *new* bidders.

no guarantee of winning the second auction, because a bidder with a higher valuation may enter later on. In case 2, the seller is better off because no bid shading occurs in the first auction, and also the final bid in the second auction becomes the expected highest valuation among *new* bidders, which is higher than the final bid without bidder entry. In case 3, the seller is better off because a new bidder wins the second auction, and the highest valuation among *new* bidders is higher than the highest valuations among the remaining *initial* bidders.

PROPOSITION 4. *By relaxing the main model to allow for bidder entry during the bidding process (and assuming product values are distributed uniformly), (1) a unique optimal degree of overlap* $\alpha^* \in [0,1]$ *exists and (2) the optimal overlap is less than that in the model without bidder entry, ceteris paribus (refer to Appendix D for the proof).*

Proposition 4 extends the results in Proposition 3 and illustrates the robustness of the results from the main model. In particular, we illustrate, similar to the results of Proposition 3, the ~~existence~~outcomes of partially overlapping strategies when new bidders enter freely during the bidding process. Bidder entry results in more bidders and bids during the auctions, which tends to increase the final price. Bidder entry, moreover, reduces the extent to which the bidder with the second~~-~~highest valuation shades her bid, because she has no guarantee that she will win the second auction, because new bidders may have a higher valuation than hers. Therefore, the optimal degree of overlap is less than that in the main model.

7. Extension: Bidders are P~~Allowed~~ermitted to Leave after the First Auction

After participating in the first auction, bidders may decide not to participate in the second auction for various reasons (e.g., they may feel they have ~~the~~ little chance to win the item in the second auction after observing the outcome of the first auction, or they may find a better alternative~~s~~ to the item or to purchasing it through the auction, or they may face~~due to~~ time constraints that prevent them from participating in the auction). Therefore, we relax the assumption in the main model that the remaining bidders in the first auction all participate in the second auction. We assume that the probability ~~for~~of a bidder ~~to~~ participat~~e~~ing in the second auction is p, where $1 \geq p > 0$, and that bidders know that they can leave during the auction, which they decide to do or not, after they finish the first auction.

Step 1: Seller's revenue.

For full overlap ($\alpha = 1$), seller's revenue remains the same as in Section 4, $E[R] = (2n-3)/(n+1) - r\sigma_{\tilde{v}}^2$.

For partial overlap ($\alpha \neq 1$), bidder behavior ~~is~~ differ~~ent~~s in compar~~ed~~ison to that in the main model. First, bidders participate in the second auction with probability p. Second, bid~~-~~ shading doesn't happen in the first auction, as the bidder with the second~~-~~-highest valuation is unsure whether she will participate in the second auction when bidding in the first auction. We solve this four-stage game (game stages are specified in section 2) via backward induction.

In **Stage 4**, we first look at bidders' expected valuation distribution in the second auction. As the probability of a bidder participating in the second auction is p, the probability of her leaving is $1-p$. Not participating is equivalent to bidding zero. As a result, $F_{(2)}(0) = 1-p$, where $F_{(2)}(.)$ denotes the CDF of bidders' expected valuation in the second auction.

Note that the posterior distribution of bidders' expected valuation without bidders' leaving is denoted as $F_{v|s}$ earlier in Section 4. Therefore, the CDF of the bidders' expected valuation in the second auction with bidders' leaving becomes

$$F_{(2)}(x) = \begin{cases} 1-p & x = 0 \\ pF_{v|s}(x) & 0 < x \leq 1 \end{cases}. \tag{20}$$

Compared to the distribution of bidders' prior expected valuation, $F_v(x)$ (the flat line on the XZ plane of Figure 2), the distribution $F_{(2)}(x)$ is still a uniform distribution, but with a jump at $x = 0$, as $F_{(2)}(0) = 1-p$.

Next we calcuate the second~~-~~-order statistics of $F_{(2)}(x)$ by replacing the distribution function in (A3) and deriving the expected mean of the 2nd highest valuation.

$$E[v_{(2)}{}^{[2]}] = (n-1)((n-1)p-1)[(2-p)\frac{(1-(1-p)^{(n-1)p})}{(n-1)p} - (1-p)\frac{(1-(1-p)^{(n-1)p-1})}{(n-1)p-1} - \frac{(1-(1-p)^{(n-1)p+1})}{(n-1)p+1}]. \quad (21)$$

As a result, the bidder with the highest valuation among all bidders wins the second auction, and the expected final bid is:

$$E[b_{(2)}^{[1]}] = E[v_{(2)}^{[2]}] - r\sigma_{\tilde{v}|s}^2 / 2, \quad (22)$$

where $\sigma_{\tilde{v}|s}^2$ is the variance of the posterior distribution of bidders' valuations.

In **Stage 3**, bidders learn and update their beliefs from what they learn by the end of the first auction. As derived earlier by Equations 16 and 18, the mean of the posterior -remains unchanged, but the variance of the posterior valuations is smaller.

$$E[E[\tilde{v}_{\boldsymbol{i}} \mid s]] = E[\tilde{v}_{\boldsymbol{i}}]. \quad (23)$$

$$\sigma_{\tilde{v}_i|s}^2 = (1-\tau)\sigma_{\tilde{v}_i}^2. \quad (24)$$

Learning changes the distribution of bidders' individual valuations $f_{\tilde{v}_i}(.)$, shown on the YZ plane in Figure 2, and the small curves of $f_{\tilde{v}_i}(.)$ s become sharper.

In **Stage 2**, all bidders bid in the first auction. There is no bid shading in the first auction, as the bidder with the second--highest valuation is unsure whether she will participate in the second auction when bidding in the first auction. -Then the final bid in the FIRST auction matches the WTP of the bidder with the second--highest valuation. That is,

$$E[b_{(1)}^{[1]}] = E[\tilde{v}_{(1)}^{[2]}] - r\sigma_{\tilde{v}}^2 / 2. \quad (25)$$

Taking the expectation on both sides of the equation above shows the individual bidder's expected valuation is equal to the mean of her product valuation, because the mean of the bidder's estimation error is zero. The means of bidders' product valuations are assumed to follow a uniform distribution in the support of [0, 1]. Then, we obtain

$$E[\tilde{v}_{(1)}^{[2]}] = E[v_{(1)}^{[2]}] = (n-1)/(n+1)\,, \tag{26}$$

thus,

$$E[b_{(1)}^{[1]}] = (n-1)/(n+1) - r\sigma_{\tilde{v}}^2/2\,. \tag{27}$$

In **Stage 1**, the seller decides the optimal overlap for maximizing his revenue $E[R] = E[b_{(1)}^{[1]}] + \beta^{1-\alpha}E[b_{(2)}^{[1]}]$. Following Equations 24 to 27, we have

$$E[R] = E[v_{(1)}^{[2]}] + \beta^{1-\alpha}E[v_{(2)}^{[2]}] - [1+\beta^{1-\alpha}(1-\tau)]r\sigma_{\tilde{v}}^2/2\,, \tag{28}$$

where $E[v_{(1)}^{[2]}]$ and $E[v_{(2)}^{[2]}]$ follow Equations 21 and 26.

Step 2: The seller's optimal overlapping strategy.

We use simulation analyses to demonstrate the seller's overlapping strategy for the different conditions in Tables WE1 and WE2. Results show that by relaxing the main model, allowing bidders to leave in the second auction, a unique optimal degree of overlap exists, where the degree of overlap is a trade ~~s~~-off between time discounting and learning. Simulation results show ~~the~~ similar patterns in the main model: ***Simultaneous auctions*** are optimal when the ease of learning (k) and valuation uncertainty ($\sigma_{\tilde{v}}^2$) are low and time discounting is high (smaller β). ***Sequential auctions*** are optimal when bidders have ample opportunity to learn because of high-valuation uncertainty and when the effect of time discounting is low (larger β). ***Partially overlapping auctions*** are optimal when valuation uncertainty is at a medium level, the learning is not too difficult, and the effect of time discounting is from medium to high, the region where the positive impact from time discounting matches the negative impact from learning. Finally, the seller's profit is larger than in the main model, due to increased learning and no bid- shading in the first auction, which dominate the negative effect from bidders' leaving in the second auction.

8. Discussion and Conclusions

Overlapping auctions are common in online auction settings. Despite the prevalence of such settings, most auctions have been modeled as stand-alone events, and bidder characteristics, such as learning and forward-looking behavior, are often ignored. This paper is motivated by a desire to better understand the popularity of overlapping auctions in the online auction environment. Toward this goal, we developed a theoretical model of a seller selling two identical products in separate concurrent auctions that captures the nature of bidding across those auctions and the bidder's ability to be forward-looking and to learn from the bidding process. Our work focuses on the (optimum degree of) overlap between the auctions. We model overlap endogenously as a function of forward-looking behavior, learning, time discounting, and varied demand (i.e., bidder entry). The combined impact of these factors determines the optimal selling format. Time discounting and forward-looking behavior favor greater overlap, whereas learning and bidders' entry favor less overlap. Therefore, partially overlapping auctions are a trade-off between these features and tend to be optimal when neither of the opposing forces dominates.

8.1. Summary of Findings

Table 4 summarizes the impact of four factors: bidders' forward-looking behavior, learning, time discounting, and varied demand. These factors influence the optimal selling strategy (degree of overlap).

We find that forward-looking bidders foresee an option to win in the second auction at a potentially lower price, resulting in bid shading in the first auction. Therefore, a seller should increase the degree of overlap to reduce bid shading. The seller's time discounting of future payoffs also has a positive effect on the degree of overlap. Hence, with forward-looking behavior and time discounting (the benchmark model), the seller's profits are always highest when conducting simultaneous auctions.

Table 4 The optimal selling strategy of different models

Models / Factor	Valuation Certainty	Valuation Uncertainty	Extension: Valuation Uncertainty & Bidder Entry	Extension: Valuation Uncertainty & Bidders Leave
Forward-looking	+	+	NA	NA

Time discounting	+	+	+	+
Varied demand	NA	NA	-	NA
Learning	NA	-	-	-
Optimal Strategy	Simultaneous Auctions	Partial overlapping exists.	Partial overlapping exists.	Partial overlapping exists.

Note: "+" ("-"): It is optimal for a seller ~~is~~ to increase (or decrease) the degree of overlap due to the specific factor.

Overlap directly influences bidder entry (the number of bidders). Therefore, reducing the overlap (i.e., a longer total duration) is optimal for the seller because~~, such that~~ more bidders can enter the auction and boost the final bid in the second auction. Learning also plays an important role in our model. When bidders are uncertain about the product value, learning helps reduce their uncertainty, resulting in more ~~-~~aggressive bidding and a higher price in the second auction. Additionally, forward-looking bidders, who are able to predict this higher future price due to learning, will bid more aggressively in the first auction, resulting in a higher price in the first auction. When bidders learn about product value at the end of the first auction, reducing the overlap is optimal for the seller, because the longer duration enhances bidder learning. Moreover, with bidder entry, less overlap leads to more bidders, and increased learning results in higher prices. Therefore, the seller wants to reduce the overlap under these conditions.

Overall, the combined impact of these factors governs the conditions for which a simultaneous, sequential, or partially overlapping strategy is optimal. When the effect of bidders' forward-looking behavior and/or the seller's time discounting dominates, running simultaneous auctions is optimal; when bidders' learning (and bidder entry) dominates, running sequential auctions is optimal. Partially overlapping auctions are optimal when neither effect dominates and when the opposing effects are mutually offsetting.

8.2. Managerial Contributions

This research has important implications for retailers selling multiple identical items through auctions over time. ~~-~~We provide important managerial guidance concerning the optimal degree of overlap under different conditions (i.e., the four factors discussed in Table 4), which has an impact on profitability.

We find that when bidders are forward-looking and when sellers discount profits, increasing the degree of auction overlap is optimal for the seller. This finding suggests sellers should increase the degree of overlap between auctions, especially for time-sensitive products such as computer products and electronics. If bidders are forward-looking, sellers should increase the degree of overlap so~~such~~ that bidders have more difficulty adjusting their bids in anticipation of future auctions (e.g., in simultaneous auctions, forward-looking behavior does not ~~exist~~occur). Alternatively, by not announcing future auctions, the seller can reduce forward-looking behavior, because bidders do not anticipate a future auction.

Information from preceding auctions provides signals about previous prices and demand, whereas information about the occurrence of future auctions provides information about supply, resulting in forward-looking behavior and bid shading. When bidders learn from the prices of previous auctions, sellers should reduce the degree of overlap, ~~such~~so that bidders have more time to learn and the variance in the information revealed is reduce~~s~~d as overlap is reduce~~s~~d and bidders have less opportunity to cross-bid. Bidder learning, ~~will~~ in particular, will be important when valuation uncertainty is high~~,~~—for example, for scarce products such as high-end jewellery or antiques, where a seller may only sell a few identical or similar items. Also, when bidders can enter the auctions over time, sequential auctions that run over a longer period of time will attract more bidders, especially in auction markets where demand is limited and sellers do not want to run multiple auctions simultaneously. We find the optimal degree of overlap increases as the number of bidders increases. Hence, sellers need to be sure to promote their auctions to obtain enough bidders when running overlapping or sequential auctions.

8.3. Limitations and Future Research

The model ~~only~~ includes bidder learning only from concurrent auctions, though bidders may also learn from the results of previously completed auctions. Websites such as eBay allow bidders to search for prices of completed auctions in the previous month. Although not all bidders will search for this information, such a search is expected to reduce the impact of learning from concurrent auctions, thus favoring increasing the degree of overlap.

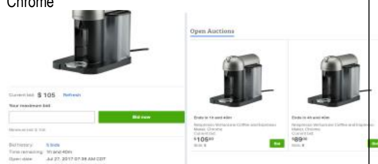
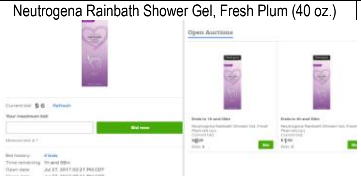
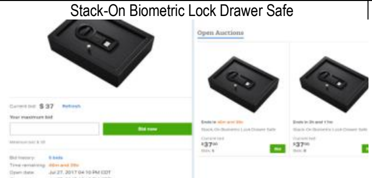
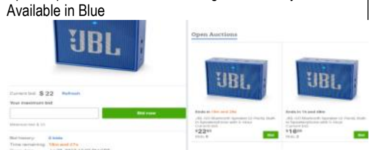
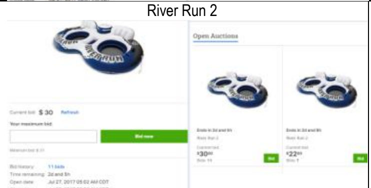
We assume a single seller participating in (up to) two auctions with unit demand. Future research should consider the influence of increased competition (i.e., more than two auctions) and bidders who have greater than unit demand (e.g., Bapna et al., 2009). An increase in the number of concurrent auctions is expected to result in more cross-bidding, which affects bidder entry and learning (Haruvy et al., 2014). Greater than unit demand should reduce bidders' forward-looking behavior (and bid shading in the first auction), behaviour which favors reducing overlap.

Commented [LG15]: Is it the behaviour or the reduction in this behaviour that favors reducing the overlap?

Our model did not explicitly consider auction duration, which may influence results (given the Poisson arrival process). However, determining the impact of duration on bidding strategies and seller profit is complex, as longer auctions may attract more bidders, but on the other hand, may deter bidders due to the longer wait or due to perceptions that competition will be higher (Muthitacharoen~~,~~ and Tams, 2017; Haruvy and Popkowski Leszczyc, 2010b). In addition, these effects may depend on other factors like the bidder pool and the type of products. We leave this as an area for future research.

Finally, we assume a single-seller platform, where the seller only considers the optimal strategy for her own auctions. On multi-seller platforms, a seller also needs to consider the strategies used by competing sellers and anticipate future strategies used by competing sellers. Future research could extend the current model to a competitive market with multiple sellers, where a seller needs to consider the (potential) impact of offers by competing sellers. Future research may also consider empirical research to study the profitability of different selling strategies. The present research can also be extended by relaxing the assumption of symmetry in bidders' responses to informative signals. Bidders may face either positive or negative signals (e.g., a selling price that is lower than expected) when updating their valuations. Also, more empirical research is needed about information provision (and bidder learning) through information overlap. What degree of overlap will result in bidder learning? Under what conditions is it best to reveal price information and in what format? Although, generally, more information is expected to result in more-aggressive bidding and higher prices (Goes, Karuga, and Tripath, 2010; Kagel and Levin, 2009), some research has found that less information may result in higher ending prices, due to some bidders overestimating the value of an item (Haruvy et al., 2014; Kagel and Levin, 1986). We may expect that a bidder's response (update) to a negative signal may be stronger than to a positive signal (e.g., Kahneman and Tversky, 1979). Future research may also integrate overlapping strategies for auctions selling complementary products with bundling across auctions (Popkowski Leszczyc and Häubl, 2010). Finally, future research may incorporate sellers' learning, because sellers may learn from the bidding behavior in the first auction, and based on this information, select the starting time of the second auction (see, e.g., Zeithammer, 2007).

Table 1: Examples of Overlapping Auctions from Sam’s Club Auction Website [a]

Overlapping Auctions		Overlapping Auctions	
Floating Island, Tahiti Island 	Duration: 8hrs Overlap: 25%	Sandusky Combination Storage Cabinet, Black 	Duration: 8hrs Overlap: 62.5%
VIZIO SmartCast 50” Class Ultra HD Home Theater Display 	Duration: 10hrs Overlap: 40%	Nespresso VertuoLine Coffee and Espresso Maker, Chrome 	Duration: 8hrs Overlap: 65%
Neutrogena Rainbath Shower Gel, Fresh Plum (40 oz.) 	Duration: 6hrs Overlap: 50%	Stack-On Biometric Lock Drawer Safe 	Duration: 6hrs Overlap: 75%
Samsung 1600 Watt Giga Sound System w/ Bluetooth 	Duration: 9hrs Overlap: 55.56%	JBL GO Bluetooth Speaker (2-Pack), Built-in Speakerphone with 5-Hour Rechargeable Battery, Available in Blue 	Duration: 8hrs Overlap: 83.33%
14K Yellow Gold Filigree Cross Pendant 	Duration: 7hrs Overlap: 57.14%	River Run 2 	Duration: 72hrs Overlap: 95.84%

[a] Data obtained from https://auctions.samsclub.com/auction/auction/, date: 2017/7/27.

Table 2. Definition of Major Symbols Used in the Paper

α	The degree of overlap between two identical product auctions
$\tilde{v}$	Bidders' product valuation, which is the true unobserved value to a bidder (v) + an estimation error ($\varepsilon_{\tilde{v}}$)
v	The true (unobserved) product value to bidders
$\tilde{v}\mid_s$	Bidders' posterior valuation of the product
$E[v_j^{[i]}]$	The expected i^{th} highest valuation among bidders in the j^{th} auction
$E[v\mid_s]$	The expected bidders' valuation after observing signal s
$\varepsilon_{\tilde{v}}$	The estimation error of the true product value
$\varepsilon_{\tilde{v}\mid_s}$	The posterior estimation error of the true product value
$\sigma^2_{\tilde{v}}$	The variance of the estimation error $\varepsilon_{\tilde{v}}$
$\sigma^2_{\tilde{v}\mid_s}$	The variance of the estimation error $\varepsilon_{\tilde{v}\mid_s}$
f_v	The density function of the true product value
$f_{\tilde{v}}(.)$	The density function of bidders' valuation of the product
$f_{\tilde{v}\mid_s}(.)$	The density function of bidders' posterior valuation of the product
F_v	The cumulative distribution function of the true product value
$F_{\tilde{v}}(.)$	The cumulative distribution function of bidders' valuation of the product
$F_{\tilde{v}\mid_s}(.)$	The cumulative distribution function of bidders' posterior valuation of the product
r	Bidders' risk coefficient
n	The number of bidders
η	Bidders' arrival rate
β	The rate at which the seller and bidders discount future payoff
Δ_α	Bid -shading when the degree of overlap is α
s	A common signal received by bidders about the product value
τ	The precision of the signal
k	The ease of learning about the product value
U	The expected utility of the bidder with the highest product valuation
R	The seller's revenue
$b_j^{[i]}$	The i^{th} highest bid in the j^{th} auction, where *j=1,2* refers to the first and second auction
WTP	Bidders' Willingness to Pay
$Initial^{[i]}$	The i^{th} highest WTP among *initial* bidders
$New^{[i]}$	The i^{th} highest WTP among *new* bidders

Table 3a The Influence of Ease of Learning (*k*) and Valuation Uncertainty ($\sigma_{\tilde{v}}^2$) on the Seller's Optimal Overlapping Strategy and Profits [a]

<table>
<tr><th colspan="10">Section 1. The Influence of Ease of Learning</th><th colspan="10">Section 2. The Influence of Valuation Uncertainty</th></tr>
<tr><th></th><th colspan="3">$\sigma_{\tilde{v}}^2$ = 0.1</th><th colspan="3">$\sigma_{\tilde{v}}^2$ = 0.5</th><th colspan="3">$\sigma_{\tilde{v}}^2$ = 0.9</th><th></th><th colspan="3">β = 0.5</th><th colspan="3">β = 0.7</th><th colspan="3">β = 0.9</th></tr>
<tr><th>k</th><th>β =.5</th><th>β =.7</th><th>β =.9</th><th>β =.5</th><th>β =.7</th><th>β =.9</th><th>β =.5</th><th>β =.7</th><th>β =.9</th><th>$\sigma_{\tilde{v}}^2$</th><th>k =.1</th><th>k =.5</th><th>k =.9</th><th>k =.1</th><th>k =.5</th><th>k =.9</th><th>k =.1</th><th>k =.5</th><th>k =.9</th></tr>
<tr><td>0.1</td><td>1[b]
1.42[c]</td><td>1
1.42</td><td>1
1.42</td><td>1
1.10</td><td>0.95
1.10</td><td>0.58
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0.85</td><td>0.1</td><td>1
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1.43</td></tr>
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1.42</td><td>1
1.42</td><td>0.98
1.42</td><td>0.96
1.11</td><td>0.86
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1.15</td><td>0
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</table>

[a] Number of bidders (n) = 11 and risk attitudes (r) = 0.8.
[b] Optimal degree of overlap, where 1 = fully overlapping (simultaneous) and 0 = no overlap (sequential). Partially overlapping auctions are highlighted in grey.
[c] The seller's optimal profits.

Table 3b The Influence of Time Discounting (β) on the Seller's Optimal Overlapping Strategy and Profits [a]

Section 3. The Influence of ***Time Discounting***									
	$\sigma_{\tilde{v}}^2$ = 0.1			*$\sigma_{\tilde{v}}^2$ = 0.5*			*$\sigma_{\tilde{v}}^2$ = 0.9*		
β	*k = 0.1*	*k = 0.5*	*k = 0.9*	*k = 0.1*	*k = 0.5*	*k = 0.9*	*k = 0.1*	*k =0.5*	*k = 0.9*
0.5	*1* [b] *1.42* [c]	*1* *1.42*	*1* *1.42*	*0.99* *1.10*	*0.83* *1.14*	*0.69* *1.21*	*0* *0.84*	*0.14* *0.99*	*0.21* *1.14*
0.6	*1* *1.42*	*1* *1.42*	*0.99* *1.42*	*0.98* *1.10*	*0.72* *1.16*	*0.53* *1.24*	*0* *0.84*	*0* *1.02*	*0* *1.19*
0.7	*1* *1.42*	*0.99* *1.42*	*0.97* *1.43*	*0.96* *1.10*	*0.52* *1.17*	*0.24* *1.28*	*0* *0.85*	*0* *1.05*	*0* *1.25*
0.8	*1* *1.42*	*0.98* *1.42*	*0.93* *1.43*	*0.90* *1.11*	*0.06* *1.21*	*0* *1.33*	*0* *0.85*	*0* *1.08*	*0* *1.31*
0.9	*1* *1.42*	*0.90* *1.43*	*0.70* *1.44*	*0.58* *1.11*	*0* *1.25*	*0* *1.40*	*0* *0.85*	*0* *1.11*	*0* *1.37*

[a] Number of bidders (n) = 11 and risk attitudes (r) = 0.8.

[b] Optimal degree of overlap, where 1 = fully overlapping (simultaneous) and 0 = no overlap (sequential). Partially overlapping auctions are highlighted in grey.

[c] The seller's optimal profits.

References

Anwar, S.; McMillan, R.; Zheng, M. 2006. “Bidding Behavior in Competing Auctions: Evidence from eBay,” *European Economic Review* (50:2), pp. 307–322.

Bajari, P.; Hortacsu, A. 2003. “The Winner's Curse, Reserve Prices, and Endogenous Entry: Empirical Insights from eBay Auctions,” *RAND Journal of Economics* (34:2), pp. 329–355.

Bapna, R.; Chang, S. A.; Goes, P.; Gupta, A. 2009. “Overlapping Online Auctions: Empirical Characterization of Bidder Strategies and Auction Prices,” *MIS Quarterly* (33:4), pp.763–783.

Bapna, R.; Goes, P.; Gupta, A.; Karuga, G. 2002. “Optimal Design of the Online Auction Channel: Analytical, Empirical, and Computational Insights,” *Decision Sciences* (33:4), pp. 557–577.

Beil, D. R.; Wein, L. M. 2009. “A Pooling Analysis of Two Simultaneous Online Auctions,” *Manufacturing and Service Operations Management* (11:1), pp. 33–51.

Betz, R., Greiner, B., Schweitzer, S. and Seifert, S., 2017. “Auction Format and Auction Sequence in Multi-item Multi-unit Auctions: An Experimental Study,” *The Economic Journal* (127:605), pp. F351–F371.

Chang, S.A. 2014. “Multiple Overlapping Online Auction Market: Bidder Strategic Mixture Analysis Using Entropy,” *Decision Support Systems* (*64*), pp.57–66.

Cramton, P.1998. “Ascending Auctions,” *European Economic Review* (42:3-5), pp. 745–756.

Elmaghraby, W.J., Gopal, A. and Pilehvar, A., 2018. “Starting Prices in Liquidation Auctions for IT Equipment: Evidence from Field Experiments,” *Production and Operations Management* (27:11), pp. 1906–1927.

Engelbrecht-Wiggans, R. 1994. “Sequential Auctions of Stochastically Equivalent Objects,” *Economic Letters* (44), pp. 87–90.

Gentzkow, M.; ~~and~~ Kamenica, E. 2011. “Competition in Persuasion,” National Bureau of Economic Research.

Goes, P. B.; Karuga, G. G.; Tripathi, A. K. 2010. “Understanding Willingness to Pay Formation of Repeat Bidders in Sequential Online Auctions,” *Information Systems Research* (21:4), pp. 907–924.

Goes, P. B.; Karuga, G. G.; Tripathi, A. K. 2012. "Bidding Behavior Evolution in Sequential Auctions: Characterization and Analysis," *MIS Quarterly* (36: 4), pp.1021–1042.

Han, J.; Qiu, C.; Popkowski Leszczyc, P. T. L. 2018. “The Effects of Competitive Reserve Prices in Online Auctions," *European Journal of Marketing* (52:7/8), pp.1439–1456.

Haruvy, E.; Popkowski Leszczyc, P.T. L.; Ma, Y. 2014. “Does Higher Transparency Lead to More Search in Online Auctions,” *Production and Operations Management* (23:2), pp. 197–209.

Haruvy, E.; Popkowski Leszczyc, P. T. L. 2018. “A Study of Bidding Behavior in Voluntary-pay Philanthropic Auctions,” *Journal of Marketing* (82:3), pp.124–141.

Haruvy, E.; Popkowski Leszczyc, P. T. L. 2010a. “Search and Choice in Online Auctions,” *Marketing Science* (29:6), pp. 1152–1164.

Haruvy, E.; Popkowski Leszczyc, P. T. L. 2010b. “The Impact of Online Auction Duration,” *Decision Analysis* (7), pp. 99–106.

Haruvy, E.; Popkowski Leszczyc, P. T. L. 2009. “Internet Auctions,” *Foundations and Trends® in Marketing* (4:1), pp. 1–75.

Hausch, D.B.1986. “Multi-Object Auction: Sequential vs. Simultaneous Sales,” *Management Science* (32:12), pp.1599–1610.Jofre-Bonet, M.; Pesendorfer, M. 2003. “Estimation of a Dynamic Auction Game,” *Economerica* (71), pp. 1443–1489.

Kagel J.H.; Levin, D. 2009. “*Common Value Auctions and the Winner's Curse*,” Princeton University Press.

Kagel J.H.; Levin, D. 1986. “The Winner’s Curse and Public Information in Common Value Auctions,”

American Economic Review (76), pp. 894–920.
Kahneman, D; Tversky, A. 1979. “Prospect Theory: An Analysis of Decision under Risk,” *Econometrica: Journal of the Econometric Society*, pp. 263–291.
Kirby, K.N. 1997. “Bidding on the Future: Evidence against Normative Discounting of Delayed Rewards,” *Journal of Experimental Psychology: General (*126), pp. 54–70.
Lange, A., List, J.A. and Price, M.K. 2011. “Auctions with Resale when Private Values are Uncertain: Evidence from the Lab and Field,” *International Journal of Industrial Organization* (*29:*1), pp.54–64.
McGee, P., 2013. “Bidding in Private-value Auctions with Uncertain Values,” *Games and Economic Behavior* (*82)*, pp.312–326.
Muthitacharoen, A., Tams, S. 2017. “The Role of Auction Duration in Bidder Strategies and Auction Prices,” *International Journal of Electronic Commerce* (21:1), pp. 67–98.
Olivola, C.Y.; Wang, S.W. 2016. "Patience Auctions: the Impact of Time vs. Money Bidding on Elicited Discount Rates," *Experimental Economics* (19:4), pp. 864–885.
Peters, M.; Severinov, S. 2006. “Internet Auctions with Many Traders,” *Journal of Economic Theory* (130:1), pp. 220–245.
Pilehvar, A., Elmaghraby, W.J. and Gopal, A., 2017. “Market Information and Bidder Heterogeneity in Secondary Market Online B2B auctions,” *Management Science* (63:5), pp.1493–1518.
Popkowski Leszczyc, P. T. L.; Häubl, G. 2010, “To Bundle or Not to Bundle: Determinants of the Profitability of Multi-Item Auctions,” *Journal of Marketing* (74:4), pp. 110–124.
Pownall, R. A.J.; Wolk L. 2013. “Bidding Behavior and Experience in Internet Auctions,” *European Economic Review* (61), pp. 14–27.
Pulley, L.1983. “Mean-Variance Approximation to Expected Logarithmic Utility,” *Operational Research* (31:4), pp. 685–696.
Roth, A. E; Ockenfels, A. 2002. “Last-Minute Bidding and the Rules for Ending Second-Price Auctions: Evidence from eBay and Amazon Auctions on the Internet,” *American economic Review* (92), pp. 1093–1103.
Wang, X.; Hu. Y. 2009. “The Effect of Experience on Internet Auction Bidding Dynamics,” *Marketing Letters (*20:3), pp. 245–261.
Wilcox, R.T. 2000. “Experts and Amateurs: The Role of Experience in Internet Auctions,”` *Marketing Letters* (11:4), pp. 363–374Zeithammer, R. 2006. “Forward-looking Bidding in Online Auctions,” *Journal of Marketing Research* (43:3), pp. 462–476.
Zeithammer, R. 2007. “Strategic Bid-Shading and Sequential Auctioning with Learning from Past Prices,” *Management Science* (53:9), pp. 1510 –1519.
Zhuang, H., Popkowski Leszczyc, P.T.L. and Lin, Y., 2018. “Why is Price Dispersion Higher Online than Offline? The Impact of Retailer Type and Shopping Risk on Price Dispersion,” *Journal of Retailing* (94:2), pp. 136–153.

Appendix A: Derivation of the seller's revenue in a single auction

***Step 1: The expectation of the Kth-highest valuation in a single auction*.**

We denote $f_v^{[k]}(.)$ and $F_v^{[k]}(.)$ as the probability distribution function (PDF) and the cumulative distribution function (CDF) of the k^{th}-highest valuation $v^{[k]}$ in a single auction, respectively. We denote the number of bidders as *n*. Then, $F_v^{[1]}(x)$ refers to the CDF of the highest valuation $v^{[1]}$ and also means the probability that all bidders' valuations are no higher than x ; that is,

$$F_v^{[1]}(x) = F_v(x)^n \ . \tag{A1}$$

We then derive its CDF:

$$f_v^{[1]}(x) = nF_v(x)^{n-1}F_v(x) . \tag{A2}$$

$F_v^{[2]}(x)$ refers to the CDF of the second-highest valuation $v^{[2]}$ and also means the probability that $v^{[2]}$ is no higher than x; that is, $F_v^{[2]}(x) = F_v(x)^n + nF_v(x)^{n-1}(1-F_v(x))$. $v^{[2]}$ is no higher than x happens in two cases: (1) All final bids are no higher than x (the first term of the RHS), or (2) $n-1$ bidders' final bids are no higher than x, but one bidder's final bid is greater than x. For the second case, there are *n* possible ways, because any bidder might be the one whose final bid is higher than x (the second term of the RHS).

Hence, the PDF of the second-highest valuation becomes $f_v^{[2]}(x) = (n-1)nF_v(x)^{n-2}(1-F_v(x))f(x)$. It follows immediately that

$$\begin{aligned} E[v^{[1]}] &= \int_{\underline{x}}^{\overline{x}} xf_v^{[1]}(x) = \int_{\underline{x}}^{\overline{x}} xnF_v(x)^{n-1}f_v(x)dx, \\ E[v^{[2]}] &= \int_{\underline{x}}^{\overline{x}} xf_v^{[2]}(x) = \int_{\underline{x}}^{\overline{x}} xn(n-1)F_v(x)^{n-2}(1-F_v(x))f_v(x)dx, \end{aligned} \tag{A3}$$

where $\underline{X}$ and $\overline{X}$ denote the lower and upper bound of the valuation distribution, respectively.

Step 2: Seller's revenue. Our analysis adopts a method similar to that in Menesez and Monteiro's paper (2005). In a single auction, bidders need to decide their best final bids, given that they know only their own valuation and the distribution of others' valuations. Bidder *i*'s utility is positive only if her final bid b_i is the highest among all bidders and thus wins the auction. In this case, her expected utility becomes

$$\begin{aligned} E[u(b_i)] &= (v_i - b_i)p_r(b_i > \max\{b_1, b_2, ..., b_{i-1}, b_{i+1}, ..., b_n\}) \\ &= (v_i - b_i)p_r(b_i > b_1, ..., b_i > b_{i-1}, b_i > b_{i+1}, ..., b_i > b_n) \\ &= (v_i - b_i)p_r(b_i > b_1)...p_r(b_i > b_{i-1})p_r(b_i > b_{i+1})...p_r(b_i > b_n) \end{aligned} .$$

Because bidders never bid over their own valuations (i.e., $b_i \le v_i$), we rewrite the above expected utility as $E[u(b_i)] = (v_i - b_i)p_r(b_i \ge v_1, ..., b_i \ge v_{i-1}, b_i \ge v_{i+1}, ..., b_i \ge v_n)$, which becomes $E[u(b_i)] = (v_i - b_i)F_v(b_i)^{n-1}$ in a symmetric equilibrium. Taking the first-order condition gives $b_i^* = \frac{(n-1)\int_{\underline{x}}^{v_{(i)}} x f_v(x) F_v(x)^{n-2} dx}{F_v(v_i)^{n-1}}$, (A4)

where b_i^* is the optimal bid of bidder *i*. The seller's revenue is the highest final bid; that is, $R = b^{[1]} = \max\{b_1^*,...,b_n^*\} = \max\{b_i^*(v_1,...,v_n)\}$.

From Equations A1 and A2, we know the probably that all bidders' valuations are below a given value v_i is $F_v(v_i)^n$, and accordingly its PDF (probability density function) becomes $nF_v(v_i)^{n-1} f_v(v_i)$. Then, the seller's expected revenue can be rewritten as $E[R] = \int_{\underline{x}}^{\bar{x}} b_i^*(v_i) n F_v(v_i)^{n-1} f_v(v_i) dv_i$. Substituting Equation A4 yields $E[R] = \int_{\underline{x}}^{\bar{x}} \frac{(n-1)\int_{\underline{x}}^{v_i} x f_v(x) F_v(x)^{n-2} dx}{F_v(v_i)^{n-1}} n F_v(v_i)^{n-1} f_v(v_i) dv_i = \int_{\underline{x}}^{\bar{x}} \left(\int_{\underline{x}}^{v_i} n(n-1) x f_v(x) F_v(x)^{n-2} dx \right) f_v(v_i) dv_i$. Further changing the order of integration (given that $\underline{x} < x < v_i$ and $v_i < \bar{x}$) and $\int_{v_i}^{\bar{x}} f_v(x) dx = 1 - F_v(v_i)$, we obtain the seller's expected revenue as:

$$E[R] = n(n-1)\int_{\underline{x}}^{\bar{x}} x(1-F_v(x)F_v(x)^{n-2} f_v(x) dx . \quad \text{(A5)}$$

Comparing Equation A3 with A5, we note that $E[R] = E[v^{[2]}]$. In a single auction, bidders drop out as the bid level increases above their valuations. Therefore, the winner is the bidder with the highest valuation among all bidders, and the winning bid rises up to $E[v^{[2]}]$, the level at which all other bidders have dropped out. As a result, the seller's expected revenue is $E[v^{[2]}]$.

In section 3, where bidders are certain about product value, bidders' valuations equal their product values. Based on Equation 1, we obtain $\tilde{v}_i = v_i$. We assume the distribution of product values follows a uniform distribution in the support of [0,1]. Then, we obtain

$$E[v^{[1]}] = \int_0^1 x n F_v(x)^{n-1} f_v(x) d(x) = \frac{n}{n+1},$$
$$E[v^{[2]}] = \int_0^1 x n(n-1)[F_v(x)^{n-2} - F_v(x)^{n-1}] f_v(x) d(x) = \frac{n-1}{n+1},$$
$$E[v^{[3]}] = \frac{n-2}{n+1}.$$

Then, the seller's expected revenue is $\frac{n-1}{n+1}$ in a single auction.■

Appendix B: Proof of Proposition 3

Step 1: Seller's revenue. We discuss the cases of $\alpha = 1$ and $\alpha \neq 1$ separately.

(1) When $\alpha = 1$ (simultaneous auctions), bidders cannot be forward-looking, and thus no bid shading occurs in the first auction. Moreover, because the two auctions end simultaneously, no learning happens. As bidders can switch between auctions, the bidder with the second-highest valuation bidder, will switch to the second auction if she is out bid in the first auction. Therefore, the highest bid level in one auction will be equal to the second-highest valuation, and the bid level in the other auction will be equal to the third-highest valuation. The seller's expected revenue, based on Equation 2, is $E[\tilde{v}^{[2]}] + E[\tilde{v}^{[3]}] - r\sigma_{\tilde{v}}^2$, which equals $E[v^{[2]}] + E[v^{[3]}] - r\sigma_{\tilde{v}}^2$, where the first two terms are derived as follows: Taking the expectation on both sides of Equation 1 yields the individual bidder's expected valuation equal to her product value, because the mean of the bidder's estimation error is zero. Moreover, we derive $E[v^{[2]}] = (n-1)/(n+1)$ and $E[v^{[3]}] = (n-2)/(n+1)$ (see Appendix A for the technical details). As product value is assumed to follow a uniform distribution in the support of [0, 1], the seller's expected revenue can be further simplified to $(2n-3)/(n+1) - r\sigma_{\tilde{v}}^2$.

(2) When $\alpha \neq 1$(partially overlapping auctions), bidders are forward-looking and learn. We solve this four-stage game (game sequence is specified in section 2) via backward induction. We first look at **Stage 4**: The remaining $n-1$ bidders bid in the second auction. The bidder with the highest valuation among all bidders in the second auction wins the product. Then, the expected final bid in the second auction is

$$E[b_{(2)}^{[1]}] = E[\tilde{v}_{(2)}^{[2]}] - r\sigma_{\tilde{v}|s}^2 / 2\,, \tag{B1}$$

where $\sigma_{\tilde{v}|s}^2$ is the variance of the posterior valuation distribution.

In **Stage 3**, as the first auction ends, bidders learn and update their beliefs. The posterior distributions are specified in Proposition 2 in the paper.

In **Stage 2**, all bidders bid in the first auction. Given that rational bidders are forward-looking, the final bid in the first auction will be influenced by bidders' bid shade. The final bid in the FIRST auction is decided by the final bid of the bidder with the second-highest valuation. With the future opportunity, she shades her final bid down to the level which makes her no difference of winning in the first or second auction. That is, $E[\tilde{v}_{(1)}^{[2]}] - E[b_{(1)}^{[1]}] = \beta^{1-\alpha}(E[\tilde{v}_{(1)}^{[2]}] - E[b_{(2)}^{[1]}])$. Substituting B1 into the equation yields

$$E[b_{(1)}^{[1]}] = (1-\beta^{1-\alpha})E[\tilde{v}_{(1)}^{[2]}] + \beta^{1-\alpha}\left(E[\tilde{v}_{(2)}^{[2]}] - r\sigma_{\tilde{v}|s}^2 / 2\right). \tag{B2}$$

We also derive $\Delta_a = \beta^{1-\alpha}(E[\tilde{v}_{(1)}^{[2]}] - E[\tilde{v}_{(2)}^{[2]}]) - r\sigma_{\tilde{v}}^2 / 2 + \beta^{1-\alpha} r\sigma_{\tilde{v}|s}^2 / 2$. Taking the expectation on both sides of (B1) shows the individual bidder's expected valuation is equal to her product value, because the mean of the bidder's estimation error is zero. Product value is assumed to follow a uniform distribution in the support of [0, 1] for simplicity. Then, we obtain: $\Delta_\alpha = \beta^{1-\alpha}/(n+1) + [(1-\tau)\beta^{1-\alpha} - 1]\sigma_{\tilde{v}}^2/2$.

In **Stage 1**, the seller decides the optimal overlap to maximize his revenue $E[R] = E[b_{(1)}^{[1]}] + \beta^{1-\alpha}E[b_{(2)}^{[1]}]$. Submitting (B1) and (B2) into the equation, we can rewrite it as $E[R] = (1-\beta^{1-\alpha})E[\tilde{v}_{(1)}^{[2]}] + 2\beta^{1-\alpha}\left(E[\tilde{v}_{(2)}^{[2]}] - r\sigma_{\tilde{v}}^2/2\right)$. Taking the expectation on both sides of Equation 1 yields the individual bidder's expected valuation equal to her product value, because the mean of the bidder's estimation error is zero. For simplicity, product value is assumed to follow a uniform distribution in the support of [0, 1]. Additionally, based on Equation 18, we obtain $E[R] = \left(n-1+\beta^{1-\alpha}(n-3)\right)/(n+1) - \beta^{1-\alpha}(1-\tau)r\sigma_{\tilde{v}}^2$.

We summarize the bid shading and the seller's expected revenue as follows:

$$\Delta_\alpha = \begin{cases} \beta^{1-\alpha}/(n+1) + [(1-\tau)\beta^{1-\alpha} - 1]\sigma_{\tilde{v}}^2/2 & 0 \le \alpha < 1 \\ 0 & \alpha = 1 \end{cases}, \tag{B3}$$

$$E[R] = \begin{cases} \left(n-1+\beta^{1-\alpha}(n-3)\right)/(n+1) - \beta^{1-\alpha}(1-\tau)r\sigma_{\tilde{v}}^2 & 0 \le \alpha < 1 \\ (2n-3)/(n+1) - r\sigma_{\tilde{v}}^2 & \alpha = 1 \end{cases}. \tag{B4}$$

Step 2: The seller's optimal overlapping strategy. The seller's optimal overlapping strategies are summarized in Lemmas 1 and 2.

Lemma 1: *When $\beta = 1$ (no time discounting), if $r\sigma_{\tilde{v}}^2 > 1/(n+1)$, then $\alpha^* = 0, E[R] = (2n-4)/(n+1) - (1-k)r\sigma_{\tilde{v}}^2$; otherwise, $\alpha^* = 1, E[R] = (2n-3)/(n+1) - r\sigma_{\tilde{v}}^2$.*

Proof: 1) $0 \le \alpha < 1$. We substitute $\beta = 1$ and $\tau = k(1-\alpha)$ into Equation B4 for the case of $0 \le \alpha < 1$. Next, taking the derivative of this revenue function yields $dE[R]/d\alpha = -rk\sigma_{\tilde{v}}^2 < 0$, showing the optimal degree of overlap $\alpha^* = 0$. The corresponding revenue $E[R|_{\alpha=0}] = E[R] = (2n-4)/(n+1) - (1-k)r\sigma_{\tilde{v}}^2$. 2) $\alpha = 1$. The revenue function (B4) is discontinuous at $\alpha = 1$, where $E[R|_{\alpha=1}] = (2n-3)/(n+1) - r\sigma_{\tilde{v}}^2$. We then compare these two revenues: $E[R|_{\alpha=0}] - E[R|_{\alpha=1}] = -1/(n+1) + kr\sigma_{\tilde{v}}^2$. This equation is positive if $r\sigma_{\tilde{v}}^2 > 1/(n+1)$. Therefore, we conclude that if $r\sigma_{\tilde{v}}^2 > 1/(n+1)$, the optimal degree of overlap $\alpha^* = 0$; otherwise, $\alpha^* = 1$ and its corresponding revenue $E[R] = (2n-3)/(n+1) - r\sigma_{\tilde{v}}^2$. ■

Lemma 2: *When time discounting exists (i.e., $\beta \ne 1$),*

1) if $\left((1-k) - (n-3)/\left((n+1)r\sigma_{\tilde{v}}^2\right)\right)\ln\beta < k$, then $\alpha^* = \tilde{\alpha}$;

2) if $\left((1-k) - (n-3)/\left((n+1)r\sigma_{\tilde{v}}^2\right)\right)\ln\beta \ge k$, then if $\beta > \tilde{\beta}$, $\alpha^* = 0$; else $\alpha^* = 1$, where $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy

3) $\left((1-k+k\alpha)-(n-3)/\left((n+1)r\sigma_{\tilde{v}}^2\right)\right)\ln\beta = k$ and $\tilde{\beta} = \left((n-2)-(n+1)(1-\beta(1-k))r\sigma_{\tilde{v}}^2\right)/(n-3)$.

Proof: When $\beta \neq 1$, first we list the necessary conditions for an optimum:

1) *FOC (First-order condition).* Substituting $\tau = k(1-\alpha)$ into the seller's revenue function in Equation B4 for the case of $0 \le \alpha < 1$ and taking the first-order derivative of the revenue function on α, we obtain $dE[R]/d\alpha = -\beta^{1-\alpha}\ln\beta\left((n-3)/(n+1)-r(1-k+k\alpha)\sigma_{\tilde{v}}^2\right)-rk\beta^{1-\alpha}\sigma_{\tilde{v}}^2$.

   Letting $\tilde{\alpha}$ be the value that makes $\frac{dR}{d\alpha}|_{\alpha=\tilde{\alpha}} = 0$, which yields the first condition:

$$\left((1-k+k\alpha)-\frac{n-3}{(n+1)r\sigma_{\tilde{v}}^2}\right)\ln\beta = k \cdot \qquad \text{(B5)}$$

The FOC is not enough to conclude $\tilde{\alpha}$ is the solution that maximizes the seller's revenue function. Therefore, we still need to check the second-order condition (SOC).

2) *SOC (Second-order condition)*

$$d^2E[R]/d\alpha^2 = (\beta^{1-\alpha}\ln\beta)[\ln\beta\left((n-3)/(n+1)-r(1-k+k\alpha)\sigma_{\tilde{v}}^2\right)+2rk\sigma_{\tilde{v}}^2]$$

Substituting Equation B5 into the SOC inequality, we obtain $d^2E[R]/d\alpha^2|_{\alpha=\tilde{\alpha}} = rk\sigma_{\tilde{v}}^2\beta^{1-\alpha}\ln\beta < 0$. (B6)

Because $d^2E[R]/d\alpha^2|_{\alpha=\tilde{\alpha}} < 0$, the derived $\tilde{\alpha}$ from the FOC is the optimal point that maximizes the seller's revenue. Next, we discuss the optimal overlap strategy under three cases.

Case 1. α^* exists in the support of (0, 1).

This case happens when both the FOC and SOC are satisfied. Because the SOC is always satisfied, we need to check the FOC:

$$\left((1-k+k\alpha)-\frac{n-3}{(n+1)r\sigma_{\tilde{v}}^2}\right)\ln\beta = k.$$

First, in FOC, because $\partial LHS(C5)/\partial\alpha = k\ln\beta < 0$, given any k, the LHS value decreases in α.

Second, because $\partial RHS(C5)/\partial\alpha = 0$, given any k, the RHS value stabilizes in α.

To fulfil FOC, the lines of LHS(B5) and RHS(B5) must intersect with each other when $\alpha \in (0,1)$.

Therefore, as long as the value LHS(B5) at the point $\alpha = 1$ is lower than the value of RHS(B5), the two lines must intersect, as in Figure B1. Thus, FOC is equivalent to $\left(-k-\frac{n-3}{(n+1)r\sigma_{\tilde{v}}^2}\right)\ln\beta < k$.

Case 2. α^* exists in the corner of [0, 1].

This case happens when FOC is not satisfied; that is, $\left(-k-\frac{n-3}{(n+1)r\sigma_{\tilde{v}}^2}\right)\ln\beta \ge k$. We calculate the revenues at these two corners to check which one is higher. Based on Equation B4, we obtain

$R|_{\alpha=1}=(2n-3)/(n+1)-r\sigma_{\tilde{v}}^2$, $R|_{\alpha=0}=\left((n-1)+\beta(n-3)\right)/(n+1)-\beta(1-k)r\sigma_{\tilde{v}}^2$, and thus

$R|_{\alpha=0}-R|_{\alpha=1}=\beta(n-3)/(n+1)-(n-2)/(n+1)+\left(1-\beta(1-k)\right)r\sigma_{\tilde{v}}^2$. Letting $\tilde{\beta}=\left((n-2)-(n+1)\left(1-\beta(1-k)\right)r\sigma_{\tilde{v}}^2\right)/(n-3)$, we conclude that if $\beta>\tilde{\beta}$, $\alpha^*=0$; otherwise, $\alpha^*=1$. ■

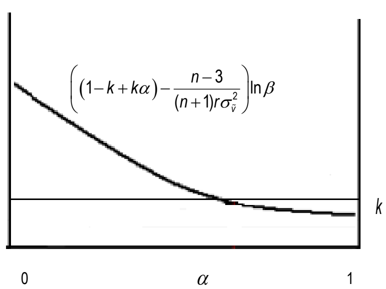

Figure B1: LHS and RHS of Equation B5

Appendix C. Optimal degree of auction overlap when bidders are risk-takers

Lemma 3: *When consumers are risk takers ($r<0$), the optimal degree of overlap is full overlap.*

Proof: Seller's profit function in Proposition 3 is $E[R]=\begin{cases}(n-1)/(n+1)+\beta^{1-\alpha}[(n-3)/(n+1)-r(1-\tau)\sigma_{\tilde{v}}^2] & 0\le\alpha<1\\ (2n-3)/(n+1)-r\sigma_{\tilde{v}}^2 & \alpha=1\end{cases}$.

Taking the first-order derivative of the revenue function with respect to α, where $0\le\alpha<1$, we obtain

$$dE[R]/d\alpha=-\beta^{1-\alpha}\ln\beta\left((n-3)/(n+1)-r\left(1-k+k\alpha\right)\sigma_{\tilde{v}}^2\right)-rk\beta^{1-\alpha}\sigma_{\tilde{v}}^2.$$

The first and the second term on the RHS are both positive when $r<0$. Therefore,

$$dE[R]/d\alpha>0 \text{ when } r<0.$$

The higher the overlap, the higher the seller's profits. Therefore, the seller will select the largest degree of overlap, that is, $\alpha\to 1$. Please note the revenue function has a jump when $\alpha=1$. Therefore, the seller's revenue function $E[R]=(2n-3)/(n+1)-r\sigma_{\tilde{v}}^2$, and the optimal degree of overlap is 1. ■

Appendix D: Proof of Proposition 4

Part 1. *Seller's optimal overlap strategies.* We discuss the cases of $\alpha=1$ and $\alpha\neq 1$ separately.

When $\alpha=1$ (simultaneous auctions), bidders are not forward-looking, and thus Δ_1 is zero as no bid shading happens in the first auction. Moreover, because the two auctions end simultaneously, no learning happens. With the average arrival rate η, the expected number of bidders participating by the end of the auctions is $N=\sum_{n=0}^{\infty}\frac{\eta^n e^{-\eta}}{n!}=\eta$. As bidders can switch between auctions, the second-highest valuation bidder, if out bid in one auction, will move to the other auction, and win that auction. Therefore, one auction ends with the final bid at the second-highest valuation, the other auction ends with the final bid at the third-highest valuation. Based on Equation 2, we derive the seller's expected revenue $E[\tilde{v}^{[2]}]+E[\tilde{v}^{[3]}]-r\sigma_{\tilde{v}}^2$, which equals $E[v^{[2]}]+E[v^{[3]}]-r\sigma_{\tilde{v}}^2$, where the first two terms are derived as follows: Taking the expectation on both sides of Equation 1 yields the individual bidder's expected valuation equal to her product value, because the mean of the bidder's estimation error is zero. Moreover, we derive $E[v^{[2]}]=(\eta-1)/(\eta+1)$ and $E[v^{[3]}]=(\eta-2)/(\eta+1)$ (see Appendix A for the technical details). As product valuations are assumed to follow a uniform distribution in [0, 1], the seller's expected revenue can be further simplified to $(2\eta-3)/(\eta+1)-r\sigma_{\tilde{v}}^2$.

When $\alpha\neq 1$(partially overlapping auctions), bidders are forward-looking and learn. We solve this four-stage game (specified in section 2) via backward induction. We first look at the **last stage**. Two segments of bidders ($\eta-1$ *initial* bidders and $\eta(1-\alpha)$ *new* bidders) bid in the second auction. With the average arrival rate η, the expected number of *initial* bidders is $N_{initial}=\sum_{n=0}^{\infty}\frac{\eta^n e^{-\eta}}{n!}=\eta$, and the expected number of *new* bidders is $N_{new}=\sum_{n=0}^{\infty}\frac{[\eta(1-\alpha)]^n e^{-\eta(1-a)}}{n!}-1=\eta(1-\alpha)$. Among them, *initial* bidders have learned information from the first auction; thus, the distributions of the two segments' valuations differ.

As before, the superscript [ ] denotes the ranking. We denote $Initial^{[i]}$ as the i^{th}-highest WTP among *initial* bidders, and $New^{[i]}$ as the i^{th}-highest WTP among *new* bidders. The superscript in ( ) also denotes whether the bidders are new or initial bidders. We derive

$$New^{[1]}=E[\tilde{v}_{(2)(new)}^{[1]}]-r\sigma_{\tilde{v}}^2/2=\eta(1-a)/\left(\eta(1-a)+1\right)-r\sigma_{\tilde{v}}^2/2,$$
$$New^{[2]}=E[\tilde{v}_{(2)(new)}^{[2]}]-r\sigma_{\tilde{v}}^2/2=\left(\eta(1-a)-1\right)/\left(\eta(1-a)+1\right)-r\sigma_{\tilde{v}}^2/2.$$

$$Initial^{[1]}=E[\tilde{v}_{(1)}^{[1]}]-r\sigma_{\tilde{v}|s}^2/2=\eta/(\eta+1)-r\sigma_{\tilde{v}|s}^2/2,$$
$$Initial^{[2]}=E[\tilde{v}_{(1)}^{[2]}]-r\sigma_{\tilde{v}|s}^2/2=(\eta-1)/(\eta+1)-r\sigma_{\tilde{v}|s}^2/2,$$
$$Initial^{[3]}=E[\tilde{v}_{(1)}^{[3]}]-r\sigma_{\tilde{v}|s}^2/2=(\eta-2)/(\eta+1)-r\sigma_{\tilde{v}|s}^2/2,$$

where $E[\tilde{v}_{(2)(new)}^{[k]}]$ denotes the expected k^{th}-highest valuation among the new bidders in the second auction.

Lemma 4. *The expected highest bid in the second auction is*

$$E[b_2^{[1]}] = \begin{cases} \max\{(\eta-2)/(\eta+1) - r\sigma_{\tilde{v}|s}^2/2,\ \eta(1-\alpha)/\left(\eta(1-\alpha)+1\right) - r\sigma_{\tilde{v}}^2/2\} & \text{won by an initial bidder} \\ (\eta-1)/(\eta+1) - r\sigma_{\tilde{v}|s}^2/2 & \text{won by a new bidder} \end{cases}.$$

Proof: Two cases are discussed. (1) **An *initial* bidder wins.** That is, when $Initial^{[2]} > New^{[1]}$ (the highest valuation among remaining initial bidders is higher than the highest valuation among new bidders). Further simplification of this condition yields $(\eta-1)/(\eta+1) - r\sigma_{\tilde{v}|s}^2/2 \geq \eta(1-\alpha)/\left(\eta(1-\alpha)+1\right) - r\sigma_{\tilde{v}}^2/2$. In this case, the expected highest bid is $\max\{Initial^{[3]}, New^{[1]}\}$, because the expected highest bid is equal to the expected second-highest valuation among all bidders in the second auction, which can be either $Initial^{[3]}$ or $New^{[1]}$. Therefore, $E[b_2^{[1]}] = \max\{(\eta-2)/(\eta+1) - r\sigma_{\tilde{v}|s}^2/2,\ \eta(1-\alpha)/\left(\eta(1-\alpha)+1\right) - r\sigma_{\tilde{v}}^2/2\}$.

(2) A *new* bidder wins. That is when $New^{[1]} > Initial^{[2]}$ (the highest valuation among new bidders is higher than the highest valuation among the remaining initial bidders). Further simplification of this condition yields $\eta(1-\alpha)/\left(\eta(1-\alpha)+1\right) - r\sigma_{\tilde{v}}^2/2 \geq (\eta-1)/(\eta+1) - r\sigma_{\tilde{v}|s}^2/2$. In this case, the expected highest bid is $Initial^{[2]}$, because the expected highest bid is equal to the expected second-highest valuation among all bidders in the second auction, which is $Initial^{[2]}$. Therefore, $E[b_2^{[1]}] = (\eta-1)/(\eta+1) - r\sigma_{\tilde{v}|s}^2/2$. ■

In **Stage three**, the first auction ends and bidders learn and update their beliefs. The posterior distributions are specified in Equations 16 and 18 in the paper.

In **Stage two**, η bidders arrive and bid in the first auction. In the previous models, the bidder with the second-highest valuation shades her final bid in the first auction. However, with the continuous arrival of new bidders, she cannot be sure whether she will be able to win in the second auction. Therefore, she does not shade her bid in the first auction, bidding up to her valuation minus her risk premium; that is,

$$E[b_{(1)}^{[1]}] = Initial^{[2]} = E[\tilde{v}_{(1)}^{[2]}] - r\sigma_{\tilde{v}}^2/2 = (\eta-1)/(\eta+1) - r\sigma_{\tilde{v}}^2/2\,. \tag{D1}$$

In **Stage one**, the seller decides the level of overlap, which maximizes his revenue. The seller's expected revenue $E[R] = E[b_{(1)}^{[1]}] + \beta^{1-\alpha} E[b_{(2)}^{[1]}]$, where $E[b_2^{[1]}]$, as analyzed in Stage 4, takes different values under different conditions (see Lemma 4).

(1) $E[b_{(2)}^{[1]}] = Initial^{[3]}$, the condition in which an *initial* bidder wins in the second auction and the expected second-highest valuation in the second auction equals the expected third-highest valuation among *initial* bidders. Then, the seller's revenue $E[R] = (\eta-1)/(\eta+1) - r\sigma_{\tilde{v}}^2/2 + \beta^{1-\alpha}\left((\eta-2)/(\eta+1) - r(1-\tau)\sigma_{\tilde{v}}^2\right)$. This happens when $Initial^{[2]} > Initial^{[3]} > New^{[1]}$.

(2) $E[b_{(2)}^{[1]}] = New^{[1]}$, the condition in which an *initial* bidder wins in the second auction and the expected

second-highest valuation in the second auction equals the expected highest valuation among *new* bidders. Then, the seller's revenue $E[R]=(\eta-1)/(\eta+1)-r\sigma_{\tilde{v}}^2/2+\beta^{1-\alpha}\left(\eta(1-\alpha)/(\eta(1-\alpha)+1)-r\sigma_{\tilde{v}}^2/2\right)$. This happens when $Initial^{[2]}>New^{[1]}>Initial^{[3]}$.

(3) $E[b_{(2)}^{[1]}]=Initial^{[2]}$, the condition in which a *new* bidder wins in the second auction and the expected second-highest valuation in the second auction equals the expected second-highest valuation among *initial* bidders. Then, the seller's revenue $E[R]=(\eta-1)/(\eta+1)-r\sigma_{\tilde{v}}^2/2+\beta^{1-\alpha}\left((\eta-1)/(\eta+1)-r(1-\tau)\sigma_{\tilde{v}}^2/2\right)$. This happens when $New^{[1]}>Initial^{[2]}>Initial^{[3]}$.

Next, based on three different revenue functions in the three cases above and following a process similar to that in step 2 of the proof of Proposition 3 in Appendix B, we derive the seller's optimal overlapping strategies.

Part 2: Compare the optimal degree of overlap with that in the main model.

Lemma 5. *The optimal degree of overlap is smaller than that in the main model.*

Proof: Compared to the main model, bidders' entry adds the impact of extra bidders, which has a negative impact on overlap but removes forward-looking behavior, which has a positive impact on overlap. As a result, the optimal overlap is less than that in the previous model without entry, ceteris paribus. ■