Chapter 1

The Maxwell equations

A short recap of the mathematical basics can be found in chapter 5.

1.1 The complete Maxwell equations

The complete set of Maxwell equations and the field quantities were introduced in the lecture "Grundgebiete der Elektrotechnik III", and the same nomenclature is generally used here. The field quantities are

$\vec{E}$ electric field strength [V/m]

$\vec{D}$ electric displacement [As/m^2]

$\vec{H}$ magnetic field strength [A/m]

$\vec{B}$ magnetic flux density [Vs/m^2]

and in general depend on the position $\vec{r}$ and time t. The sources of the electric and magnetic fields are

ρ space charge density [As/m^3]

$\vec{J}$ electric current density [A/m^2]

1.1.1 The Maxwell equations in integral form

It will be assumed that all boundary curves C and surfaces F are static, and relativistic effects can thus be ignored.

1st Maxwell equation (Ampère's circuital law, field equation for $\vec{H}$)

$$\oint_C \vec{H} \cdot m_d\vec{r} = \int_F \left(\vec{J} + \frac{\partial}{\partial t} \vec{D} \right) \cdot m_d\vec{F} \tag{1.1}$$

Figure 1.1: Law of flux

Figure 1.2: The fourth Maxwell equation.

2nd Maxwell equation (law of induction, field equation for $\vec{E}$)

$$\oint_C \vec{E} \cdot m_d\vec{r} \;=\; - \int_F \frac{\partial}{\partial t} \vec{B} \cdot m_d\vec{F} \tag{1.2}$$

3rd Maxwell equation (field equation for $\vec{D}$)

$$\oint_F \vec{D} \cdot m_d\vec{F} \;=\; \int_V \rho \, m_dV \tag{1.3}$$

4th Maxwell equation (field equation for $\vec{B}$)

$$\oint_F \vec{B} \cdot m_d\vec{F} \;=\; 0 \tag{1.4}$$

The closed boundary curve C encloses the area F, with the two positively oriented towards each other in the mathematical sense. The surface F fully encloses the volume V and is oriented outwards.

1.1.2 The constitutive relations in static matter

The constitutive relations with electric field constant ($\varepsilon_0 = 8.854187817 \cdot 10^{-12} \frac{\mathrm{As}}{\mathrm{Vm}}$) and magnetic field constant ($\mu_0 = 4\pi \cdot 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2}$) in a vacuum are

$$\vec{D} \;=\; \varepsilon_0 \, \vec{E} \;+\; \vec{P} \tag{1.5}$$

$$\vec{B} \;=\; \mu_0 \left(\vec{H} \;+\; \vec{M} \right) \tag{1.6}$$

and the values defined in matter

- $\vec{P}$ electric polarisation [As/m^2]
- $\vec{M}$ magnetisation [A/m] .

(1.1) - (1.6) are always valid in a vacuum; in the presence of matter, strictly speaking they hold only as long as the field values do not vary much with the distances between the atoms or molecules ($\longrightarrow$ electron theory)

No relationship analogous to (1.5), (1.6) with comparable generality exists for $\vec{E}$ and $\vec{J}$. However, for many technically important, static, linear, isotropic fixed bodies, the following applies as a close approximation:

$$\vec{J} \;=\; \sigma \left(\vec{E} \;+\; \vec{E}^{(}e) \right) \quad . \tag{1.7}$$

$\vec{E}^{(}e)$ here captures current density components whose occurrence is not due to the dynamic effect of the electrical field strength $\vec{E}$. For example, these could be diffusion currents in semiconductive components. σ is the conductivity of the material.

1.1.3 The Maxwell equations in differential form and the corresponding interface conditions

Should there be no singularities other than interfaces, such as e.g. point charges, linear charges or linear currents, then the Maxwell equations in integral form (1.1) - (1.4) lead directly to the validity of the Maxwell equations in differential form and the associated interface conditions.

1st Maxwell equation in differential form and associated interface conditions

$$\begin{aligned} \text{rot}\ \vec{H} &= \vec{J} + \tfrac{\partial}{\partial t}\ \vec{D} && \text{apart from interfaces} \\ \text{Rot}\ \vec{H} &= \vec{J}_F && \text{at the interfaces} \end{aligned} \tag{1.8}$$

($\vec{J}_F$ is the surface current density (linear current density) with units [A/m]. Except for a special case covered later, in static matter $\vec{J}_F = \vec{N}$ always holds for the surface current.)

2nd Maxwell equation in differential form and associated interface conditions

$$\begin{aligned} \text{rot}\ \vec{E} &= -\tfrac{\partial}{\partial t}\ \vec{B} && \text{apart from interfaces} \\ \text{Rot}\ \vec{E} &= \vec{N} && \text{at the interfaces} \end{aligned} \tag{1.9}$$

3rd Maxwell equation in differential form and associated interface conditions

$$\begin{aligned} \text{div}\ \vec{D} &= \rho && \text{apart from interfaces} \\ \text{Div}\ \vec{D} &= \rho_F && \text{at the interfaces} \end{aligned} \tag{1.10}$$

(ρ_F is the surface charge density with units [As/m^2]. In this course, it will only be taken into account for conductors.)

4th Maxwell equation in differential form and associated interface conditions

$$\begin{aligned} \text{div}\ \vec{B} &= 0 && \text{apart from interfaces} \\ \text{Div}\ \vec{B} &= 0 && \text{at the interfaces} \end{aligned} \tag{1.11}$$

Inverting these under the conditions given in (1.8) - (1.11) in all spatial points immediately yields in turn the validity of (1.1) - (1.4) for arbitrary curves C enclosing areas F and arbitrary volumes V with surface areas F. Thus in this respect, (1.1) - (1.4) and (1.8) - (1.11) are equivalent.

The definitions of surface divergence Div (5.45) and surface rotation Rot (5.46) are given in Section 5.4. The surface charge density ρ_F and the surface current density $\vec{J}_F$ are ideals; in reality these are very high densities in very thin layers. The ideal interfaces stand for constant changes in matter, often taking place within a few atomic layers through abrupt transitions. Making such idealised assumptions simplifies the problem and is very often justifiable.

It should furthermore be noted that the Maxwell equations are not completely independent of each other. If we apply the divergence from (1.9) to both sides of the equals sign, then for sufficiently smooth fields

$$\operatorname{div}\left(\operatorname{rot}\ \vec{E}\right) = \nabla\cdot\left(\nabla\times\vec{E}\right) = (\nabla\times\nabla)\cdot\vec{E} = 0 = \operatorname{div}\left(-\frac{\partial}{\partial t}\ \vec{B}\right) = -\frac{\partial}{\partial t}\operatorname{div}\ \vec{B}\,. \tag{1.12}$$

This, however, corresponds precisely to the negative temporal derivative of (1.11).

1.1.4 The continuity equation

Taking the divergence of the first Maxwell equation (1.8) and using (1.10), we obtain the continuity equation

$$\operatorname{div}\left(\operatorname{rot}\ \vec{H}\right) = \nabla\cdot\left(\nabla\times\vec{H}\right) = (\nabla\times\nabla)\cdot\vec{H} = 0 = \operatorname{div}\left(\vec{J}+\frac{\partial}{\partial t}\vec{D}\right) = \operatorname{div}\vec{J}+\frac{\partial\rho}{\partial t} \tag{1.13}$$

and thus

$$\operatorname{div}\vec{J}+\frac{\partial\rho}{\partial t} = 0\,. \tag{1.14}$$

The corresponding interface condition is

$$\operatorname{Div}\vec{J}+\frac{\partial\rho_F}{\partial t} = 0\,. \tag{1.15}$$

1.2 The energy theorem

The scalar multiplication of (1.8) with $\vec{E}$ and (1.9) with $\vec{H}$ away from the interfaces yields, after differentiation,

$$\vec{E}\cdot\operatorname{rot}\ \vec{H} - \vec{H}\cdot\operatorname{rot}\ \vec{E} = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial}{\partial t}\ \vec{D} + \vec{H}\cdot\frac{\partial}{\partial t}\ \vec{B}\,. \tag{1.16}$$

Taking (5.53), we obtain

$$\begin{aligned}\operatorname{div}\left(\vec{E}\times\vec{H}\right) = \nabla\cdot\left(\vec{E}\times\vec{H}\right) &= \nabla\cdot\left(\overset{\downarrow}{\vec{E}}\times\vec{H}\right) + \nabla\cdot\left(\vec{E}\times\overset{\downarrow}{\vec{H}}\right)\\ &= \vec{H}\cdot\left(\nabla\times\vec{E}\right) - \vec{E}\cdot\left(\nabla\times\vec{H}\right)\\ &= \vec{H}\cdot\operatorname{rot}\ \vec{E} - \vec{E}\cdot\operatorname{rot}\ \vec{H}\,,\end{aligned} \tag{1.17}$$

where the arrows indicate which value in the bracketed expression the nabla operator is applied to.

Overall, it thus follows from (1.16), (1.17) away from the interfaces, that

$$-\operatorname{div}\left(\vec{E}\times\vec{H}\right) = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial}{\partial t}\ \vec{D} + \vec{H}\cdot\frac{\partial}{\partial t}\ \vec{B}\,. \tag{1.18}$$