**Introduction**

Integro-differential equations appear very naturally in various applications (see for example [9,10,17,18,25]), which explains the interest in the theory of these equations (see for example [2,3]). Such systems are found in models for many mechanical systems.

Consider the non-linear system of integro-differential equations:

where

 is an n-dimensional vector

are constant matrices

is a symmetric positive-definite matrices.

Let us define the matrices and as

where are symmetric and are skew symmetric matrices. We will use terminology common in mechanics, describing a classification of forces acting on the system:

1. Potential force =
2. Dissipative force =
3. Gyroscopic force =
4. Bounded damping forces =
5. Non-linear force = .

In the absence of the non-linear force , the stability and oscillation of the system (1) are well studied. The direct Lyapunov method has been usefully employed for investigating the stability of ODE systems, when the linear approximation is non critical.

For non-linear force X we have the following result:

**Theorem**

 Let the system

where

 are vector-polynomials of degree in .

Let the system be non-resonant, i.e., for any integer-valued vector

.

the zero solution of (0.1a) is Birkhoff stable (stable in any finite non-linear approximation).

If only potential and non-linear forces are present in (1) it is reducible to

where

.

The spectrum of linear approximation has n couple of pure imaginary eigenvalues. Such a system is primarily used in the theory of non-linear oscillations in which we realize the critical case of stability.

In this paper we want investigate the stability of integro-differential systems with non-linear force when the entire spectrum, or it’s non-linear approximation part, is located on the imaginary axis.

First, we need reduce the integro-differential system to the corresponding system of ODE. This can be found in the first part of this paper. The idea of a reduction to a system of ordinary differential equations in the study of stability was presented in [5]. In the second part of this article we will present the method and definitions relative to reducing non-linear systems as much as possible to a simpler form. We refer to this method as normalization, and to the form of the system after this method as being reduced to normal form, or simply n.f.

In the third part of the article we will find the conditions for stability and instability of the zero solution of a system reduced to normal form.

Finally, in the fourth part of this article, we will show how it is possible to use additive integrals as a stability control. This means that if the solution of the non-linear oscillator is unstable, then it is possible to choose coefficients of the additive integral that will make the solution stable or vice versa.

**Part 1 -Reduction method**

The idea and development of the reduction method for integro-differential equations was described in the works of Domoshnitsky and Goltser[5,6 ].

We will present how it will be possible to reduce a system of integro-differential equations to a system of ODEs.

Let us consider a system of integro-differential equations:

Let us assume that

.

Then the system (1.1) can be written in the form

If the kernel is a square integrable function, then in a Hilbert space it can be represented in the form:

where

 are matrices continuous on . If we assume that are reversible matrices we can write in the following form

and assume that

Let us denote

where is in the form (1.4).

Let us introduce a new variable

where

,

Then system (6) can be reduced to a system of ODEs

where

 .

This raises the question under what conditions can we obtain a matrix with constant coefficients? We will now find sufficient and necessary conditions where matrix can be reduced to such a matrix.

Let be FSS and assume that we can write in the following form:

where

 = Lyapunov matrix

 = constant matrix.

Substitute (1.5) in the system:

 and we get

.

**Theorem**

 The system of equations

is reducible to a sysem with constant coefficients

if

.

This condition is necessary and sufficient if we use Erugin’s results on reducible systems:

**Theorem [Erugin]**

The linear differential system

is reducible if and only if some part of its fundamental matrix *Y(t)* can be represented by a Lyapunov matrix *L(t)* multiplied by the exponential of the product of the independent variable *t* by the constant matrix *B*. That is:

.

We now consider several cases.

(а) is Cauchy, in which case we denote

such that is a Cauchy function of equation:

Since is a Cauchy function it has the following properties:

where

 is the order of the linear differential operator *P*, which is determined by the structure of .

Thus, from system (1.1) we derived the following 2n dimension system of ODE

If *K(t,s)* is not a Cauchy function, refer to the next case.

(b) Leontief functions are a rather broad class of functions that allows an expansion in the form of a generalized Dirichlet series. Let the function have the following representation :

.

We will examine the systems whose kernel is a difference equation.

Assume that is an analytical function and can be written in the form

.

Then can be developed into a series (1.6) of exponents (see, Leontieff)

From (1.6) can be learned

1. There is an option for countable system of ODEs.
2. Each exponent is a Cauchy function of some ODE.
3. These equations are different depend on whether is real or complex.

There is another way to expand *K(t-s)* as a Fourier series; for example, as in the following:

.

We can substitute

and get the necessary expansion.

(c) Periodic case.

Assume that is an -periodic matrix. From the equation

using Floquet's theory, we come to the conclusion that in this case the kernel contains the matrix

where is -periodic with respect to .

Thus, using Floquet’s theorem as a frequent case of the theorem of reducible systems, it can be reduced to a system with constant coefficients.

In this example, we want to use a kernal with a periodic form.

Example

As in the first example, we will consider the application of the reduction method to the study of non-linear oscillators with non-linear force represented by an integral.

Let's consider system (1) where , ,

Such a system will look like this

.

 It is not difficult to see that the operator *P* has the form

 with second order operator.

After using the reduction method, the system looks like this

**Part 2 -Normal Form**

Let us briefly consider the idea of reducing the system to its normal form. For this we will use terminology which can be found in Bibikov [1].

 Consider two formal systems of ordinary differential equations

and

where

 are formal power series.

Definition

We say that systems (2.1) and (2.2) are *formally equivalent* if there exists a change of variables

where

is a formal power series, which reduces (2.1) to (2.2).

Let be the vector whose co-coordinates are eigenvalues of matrix .

Theorem

 If

 then system (2.1) is formally equivalent to any system (2.2) and in (2.3) is uniquely determined.

We seek the simplest form of such a system, it is convenient to assume that is a Jordan canonical matrix. This can be achieved by means of linear singular changes of variables.

 So we consider a system:

Definition

 Considering system (2.5) we say that coefficients of any power series corresponding to pair satisfying

are resonant and the corresponding terms are called resonant terms. On the other hand, If

we say that the coefficients and corresponding terms are non-resonant. Equation (2.6) is called a resonance equation.

Definition

 System (2.5) where all non-resonant terms are equal to zero is called a normal form (NF).

As an example of finding the normal form let’s consider a system of non-linear oscillators where, for example, the first oscillator is disturbed by a force represented by the integral:

Using the reduction method described above we obtain the following system of equations:

If we make the following substitutions

then system (2.8) in matrix form can be written in the following manner:

where

We now substitute the variable in order to put the matrix *A* in diagonal form

and substituting this back into (2.9) we get

If we multiply the last equation on the left by we get

and using the new notation we have

where

 is a new non-linearity

Making the following substitution

after which we want to obtain

Substitute (2.11) in (2.10) and we get

Using expression (2.12):

After simplifying we then get

where

is obtained after substituting row by row. Equation (2.13) is called a homological equation.

Now we need to investigate the case when equation (2.13) has a solution for .

Consider equation *s* and the equation for finding the coefficients of the terms of form j of order

and we get

The last condition will help us find all resonance values of vector *p*. The result we get can be formulated with the following theorem.

**Теорема**

 The system of equations

using the following substitution

reduces to the form

where

 - has only resonant terms, which can be found using the formula

Example – normal form

Using condition (2.8) in a system of two pairs of imaginary eigenvalues in order to find the resonant terms remaining after the normalization method:

then the first non-linear terms would be and .

then the first non-linear terms would be and .

As we saw above, second order terms in the absence of external resonance do not influence the structure of the normal form and we can choose them to be equal to zero. The non-linearities, then, can be chosen as follows:

Let's start the normalization after reduction of the integro-differential equation to a system of ODEs, so consider the next system of equations

Make the change of variables (2.15)

and we get the following system of ODEs

This system can be written in the following form:

or more concisely, it can be written as

The eigenvalues of matrix A will be

After transforming the linear part of system (2.16 ) to diagonal form using linear transform , , where transform matrix and its inverse transform have the form :

substitute this change into (2.16) to get

.

Multiply this on the left by and we have

Using the new notation, we get

where

 is a new non-linearity obtained after row by row substitution.

We now make a near identical substitution

and we now want to get

Coefficients, which must be found in formal form, will be

.

We will write here only the final result for the coefficients of the normal form:

The normal form up to the 3rd order have the following form

**Теорема**

 The following system of equations:

is reduced after normalization to the form:

The coefficients of the normal form are found in the equality (2.17).

**Part 3 - Investigating the stability of the zero solution to the equation ( )**

System (2.18) was examined in the works of Lyapunov, Malkina, Veretennikov, Molchanov, and Goltser. We will use part of the results obtained in these studies to examine the stability of the zero solution of this system.

Consider the following system

Making the substitutions

we obtain

Equating the real and imaginary parts we get

Or, we can write it as

The following theorem can provide us with an answer to the stability of the solution of the preceeding system.

**Theorem** [Goltser ]

1. For asymptotic stability of the solution of the system

 Regardless of the members , it is necessary and sufficient that , and that one the following two conditions should be fulfilled:

 For :

1. If the solution for the system ( ) is asymptotically stable, regardless of the members , then for the system ( ) there exist Lyapunov function , , with the sign-definite derivative, by virtue of the system ( ).

Applying the last theorem to our example, we get

or

In order for the zero solution to be asymptotically stable, the following conditions must be met:

or

Analyzing conditions () we get

**Theorem**

 In order for the zero solution of the system

to be asymptotically stable the following inequality must be met

Note:

The previous inequality imposes a condition on the coefficient such that the solution to the equation with integral addition is stable, but this means that and under this condition the solution of the equation without integral addition will be stable.

**Part 4 - The use of integral addition as a stability control of a non-linear oscillator.**

Let the following equation be given

such that and the zero solution is not stable.

Using integral addition of the following form

we want to obtain stability of the zero solution of the new equation

.

After using the reduction principle we get the following system of equations

Employing a change of coordinates we get the following system of equations

.

The structure of the normal form will be the same so, as before, we will write out only the final coefficients of the normal form:

where

Making a change of coordinates we get the following system:

Using theorem () we need to check the following inequality:

To simplify, we will let

then we have

The last inequality can be replaced by the following inequality:

In each inequality we find an expression for

.

Let’s assume that

then we can write

or

It follows that if we choose such that the last inequality ( ) is true, then we can choose such that the inequality () is also true.

**Theorem**

 The zero solution to the equation

will be asymptotically stable if the following conditions are met

.

**Reference**

[1] Ju.N. Bibikov, Local Theory of Nonlinear Analytic ODEs, in: Lecture Notes in Mathematics,vol. 702, Springer-Verlag, 1979

[2]C.Corduneanu, Integral Equations and Stability Feedback Systems.-New York, London: Academic Press,1973.

[3] C.Corduneanu, Integral Equations and Applications ,Cambridge University Press, Campbridge, 1991

 [4] A.D. Brjuno, The Local Method of Nonlinear Analysis, Nauka, Moscow, 1972 (in Russian)

[5] A. Domoshnitsky. Exponential stability of convolution integro-differential equations, Functional-Differential Equations, 1998, vol.5, pp.445-455.

 [6] A.Domoshnitsky, Ya. Goltser, One approach to study of stability of IDE, Nonlinear Analysis 47 (2001) 3885-3896.

[7] A.Domoshnitsky, Ya. Goltser, Hopf bifurcations of IDEs, Qualitative Theory of Differential Equations 3 (2000) 1-11.

[8] A.Domoshnitsky, Ya. Goltser, Floquet theorem and stability of linear integro-differential equations, Functional Differential Equations 10(3-4) (2003) 463-471.

[9] A.D. Drozdov and V.B. Kolmanovskii, Stability in Viscoelasticity, North Holland, Amsterdam, 1994.

[10] M.Fabrizio and A,Morro. Mathematical Problems in Linear Viscoelasticity, SIAM Stud.Appl.Math., Philadelphia, 1992.

[11] F.R. Gantmacher, Theory of Matrices, Nauka,1967.

[12] Ya. Goltser, The process of normalization and solutions of bifurcations problems of the oscillations and stability theory: A synopsis, Functional Differential Equations 1 (1993) 108-125.

[13] Ya. Goltser, On the stability of differential equations systems with the spectrum on the imaginary axis, Functional Differential Equations 4(1-2) (1997) 47-63.

[14] Ya. Goltser, A.Domoshnitsky, Bifurcation and stability of IDEs, Nonlinear Analysis 47(2001) 953-967.

[15] Ya.Golster, A.L.Kunitsyn, On the stability of autonomous system with internal resonance, Prikladnay Matematika I Mekhanika 59(6) (1075) 974-985(Russian). English trans.:Journal of App;ied Mathematics and Mechanics, PMM,39(6)(1976).

[16] Ya.Goltser, E.Litsyn, Volterra integro-differential equations and infinite systems of ODEs,Mathematical and Computer Modeling 42(2005) 221-233.

[17] J.M.Golden and G.A.Graham. Boundary value problems in linear viscoelasticity. Springer-Verlag,1998.

[18] M.E.Gurtin and Pipkin, A General Theory of Heat conduction with FiniteWare Speeds, Arch. Rat.Mech.Anal. 31(1968) 113-

[19] G.V.Kamenkov, selected work,vol. 1, 1979;vol.2,1972.

[20] A.M. Lyapunov, Collected work,vol. 2, Moscow, 1956.

[21] I.G. Malkin, Theory of Movement Stability, Nauka, 1966 (in Russian). Translation by the Atomic Energy Association, AGE-TR.-3352.

[22] I.G. Malkin,Nekotorye zadachi nelinejnykh kolebanij, 2004

[23] D.R. Merkin, Introduction to the Theory of Movement Stability, Nauka, 1971 (in Russian)

[24] A.M. Molchanov, Stability and Oscillations of Nonlinear approximation, Doklady Akademii Nauk SSSR 141 (1967) 24-27.

[25] A.Novick-Cohen. Conserved Phase-Field Equations with Memory, in Curvature Flows and Related topics, A. Damlamian, J.Spruck and A.Vistin, eds.,pp. 179-197, GAKUTO Internat.Ser. Math. Sci. Appl.,5. Gakkotosho, Tokyo, 1995.

[26] V.G. Veretnikov, Stability and Oscillations of nonlinear Systems, Nauka, Moscow, 1984 (in Russian).

[27] V.R. Rumyantsev, A.S. Oziraner, Stability and Motion Stabilization for part of variables, Nauka, Moscow, 1987 (in Russian).